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TECHNICAL NOTE

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OPTIMAL FILTERING AND LINEAR PREDICTION APPLIED

TO A MIDCOURSE NAVIGATION SYSTEM

FOR THE CIRCUMLUNAR MISSION

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TABLE OF CONTENTS

	Page
SUMMARY	1
INTRODUCTION	2
NOTATION	2
Superscripts	4
Subscripts	4
THE NAVIGATION SYSTEM	5
General Description	5
Theoretical Considerations	7
The optimal filter	7
THE DIGITAL COMPUTER SIMULATION	10
Description of the System	10
Assumptions of System Characteristics	11
Optical instrumentation	11
Sequence of observations	12
Velocity corrections	13
Injection errors	13
The standard case	13
Calculation of Statistical Information	13
RESULTS	14
Errors Due to Linear Prediction	15
Operation of the Navigation System	18
Comparison of statistical data with one ensemble member	18
Estimation errors	19
Prediction errors	19
Indicated velocity correction	20
Deviations from the reference trajectory	21
Terminal conditions	21
Computer requirements	24
CONCLUDING REMARKS	24
APPENDIX A - NONLINEAR EQUATIONS OF MOTION FOR TRAJECTORY CALCULATIONS	26
APPENDIX B - CALCULATION OF TRANSITION AND PREDICTION MATRICES	29
APPENDIX C - CALCULATION OF MEASURED ANGLES	36
APPENDIX D - COMPUTATION OF STATISTICAL INFORMATION	38
REFERENCES	45
TABLES	46
FIGURES	47

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SUMMARY

A midcourse navigation system for the circumlunar mission has been designed and simulated on a digital computer. The spacecraft's actual position and velocity are estimated by means of the optimal filter described in NASA TN D-1205. On-board optical measurements of the subtended earth and moon angles and the angular positions of the earth and moon centers relative to an on-board reference system are assumed as the basic measurements. Linear prediction is then utilized to compute the corrective velocity impulse necessary to reduce the estimated position deviation at the end point to zero for a fixed time of arrival.

The mission is divided into two phases, outbound and return. The objective of the outbound phase is to arrive at a prescribed perilune point at a fixed time. Terminal conditions at this time are taken then as initial conditions for the return flight to a prescribed perigee point in the center of the earth's re-entry corridor.

The results of a digital computer simulation of the navigation system are presented. The simulation of the trajectory is three-dimensional and includes the gravitational effects of the earth (through the second harmonic term) the moon, and the sun, and random errors in injection conditions, optical measurements, and application of corrective velocity impulses.

The effects of variations in the root-mean-square optical measurement and injection errors on the deviation in end position and cumulative fuel consumption for velocity corrections during the mission are given. In addition, some results are presented which give a quantitative idea of the inaccuracies in linear prediction resulting from neglecting nonlinearities.

INTRODUCTION

Reference 1 described how some advanced concepts of statistical linear filter theory proposed by R. E. Kalman (ref. 2) could be used to obtain the filter which would give the best estimate of the trajectory (position and velocity vectors) of a space vehicle. Data were given also on the performance of this optimal filter for accuracies judged achievable by on-board optical instruments.

The safety of a manned circumlunar mission is increased if an on-board midcourse navigation system does not have to rely on tracking information obtained on earth; it is apparent that the use of an optimal filter in such a system should be explored. It is the purpose of this report, therefore, to present details of a possible on-board navigation system that uses the optimal filter and to evaluate the performance of the system. The guidance system chosen is based on the assumption that the actual trajectory of the space vehicle can be accurately described in terms of linear perturbations about a precalculated standard or reference trajectory. This technique has been used successfully in many problems involving the dynamics of aircraft and, because of its simplicity, could prove useful in space navigation problems.

NOTATION

A	prediction matrix (relating deviations from the reference trajectory at the end point to those at earlier times)
C	transformation matrix for converting errors in angles and magnitude of velocity correction into Cartesian coordinates
E	expected value
F	matrix of coefficients for linear differential perturbation equations
I	unit matrix
K	weighting matrix
P	covariance matrix of estimation errors

$\tilde{p}$	magnitude of error in predicting position deviations at the end points
R	covariance matrix of deviations between actual and reference trajectories
R_e	range of vehicle from earth
R_m	range of moon from earth
R_s	range of sun from earth
r	magnitude of position deviation between actual and reference trajectories
$\tilde{r}$	magnitude of position deviation between actual and estimated trajectories
t	time
u	magnitude of velocity correction
v	magnitude of velocity deviation between actual and reference trajectories
$\tilde{v}$	magnitude of velocity deviation between actual and estimated trajectories
X, Y, Z	Cartesian coordinates of vehicle's position
X_m, Y_m, Z_m	Cartesian coordinates of moon's position
X_s, Y_s, Z_s	Cartesian coordinates of sun's position
x	state vector (6×1 matrix of vehicle's position and velocity deviations from reference)
$\hat{x}$	estimated value of x
$\tilde{x}$	error in estimating x
x_i	component of x
x_c	value of x immediately after a velocity correction
$\bar{x}$	3×1 matrix of position components of x
$\dot{\bar{x}}$	3×1 matrix of velocity components of x

α	elevation angle of earth or moon
β	azimuth angle of earth or moon
γ	one-half the subtended angle of earth or moon
δ	small deviation
σ	standard deviation of angular measurement errors
θ	elevation angle of corrective velocity vector
ψ	azimuth angle of corrective velocity vector
Φ	transition matrix

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Superscripts

T	transpose of a matrix
$(\dot{\ })$	derivative with respect to time
$(\bar{\ })$	3×1 matrix

Subscripts

E	end point
G	velocity to be gained
Q	error in velocity correction
e	earth
i,k	integers
m	moon
rms	root-mean-square value
s	sun

THE NAVIGATION SYSTEM

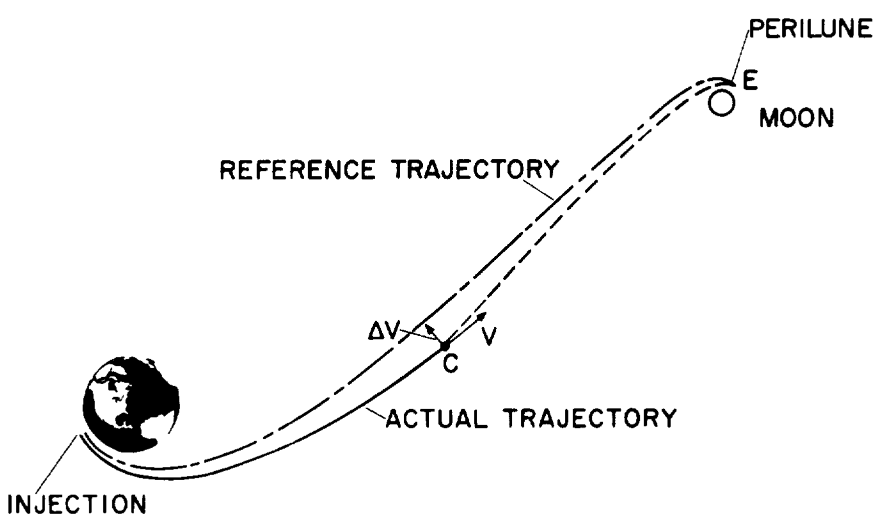
General Description

The ground rules postulated for this study were as follows:

1. Manned reconnaissance of the moon is the principal objective (in the present case by means of a single passage around the moon). Thus, accuracy in achieving a prescribed perilune and, more importantly, a prescribed perigee for a safe return to earth is required with a minimum expenditure of fuel for application of corrective velocities.

2. To provide increased safety for the crew of the vehicle, the midcourse guidance system shall use on-board instrumentation and computations to provide navigation information, but shall also have the capability of accepting guidance information from other sources (for example, ground tracking data).

The design of a midcourse navigation system fulfilling these ground rules is a control problem like many others, the aspects of which may be described as follows. Sketch (a) illustrates the first half of the midcourse lunar flight. Suppose that errors at injection cause the vehicle to depart from the reference trajectory so that desired end-point conditions (e.g., at perilune, E) will not be achieved. Suppose further that at point C a velocity increment, ΔV, is to be added to correct for these errors. First, it is necessary to determine by means of data



Sketch (a)

obtained from imperfect sensors (optical instruments are assumed here) as good an estimate as possible of the position and velocity (i.e., the state) of the vehicle at point C. This might be called trajectory determination. Then, on the basis of the best estimate of the trajectory, end point conditions must be predicted (e.g., what will be the estimated perilune, point E, if no corrective action is taken). Next, a guidance law must be used which makes possible calculation of desired corrective action to change the estimated end point conditions to correspond to those desired. Finally, the indicated control action must be implemented by applying thrust.

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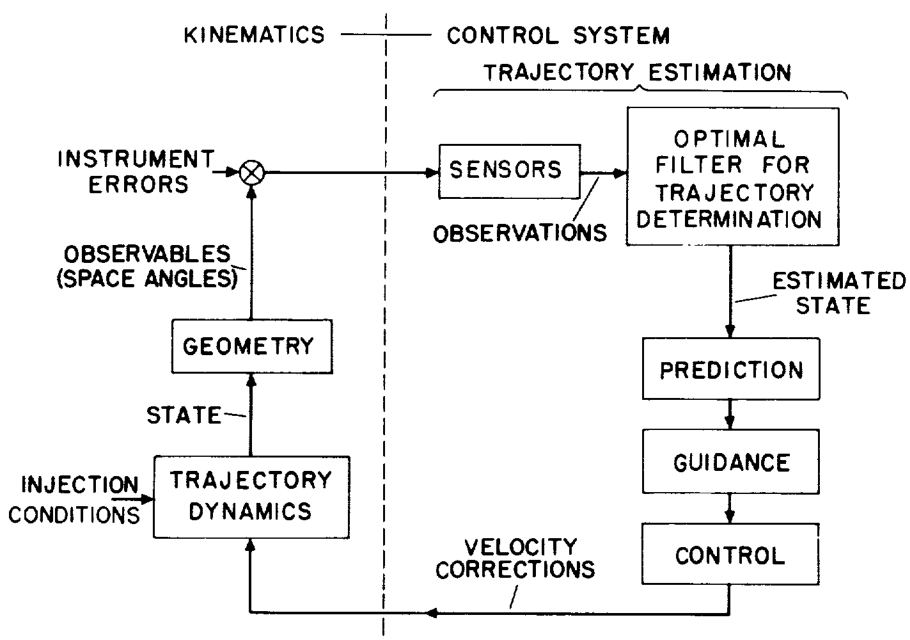
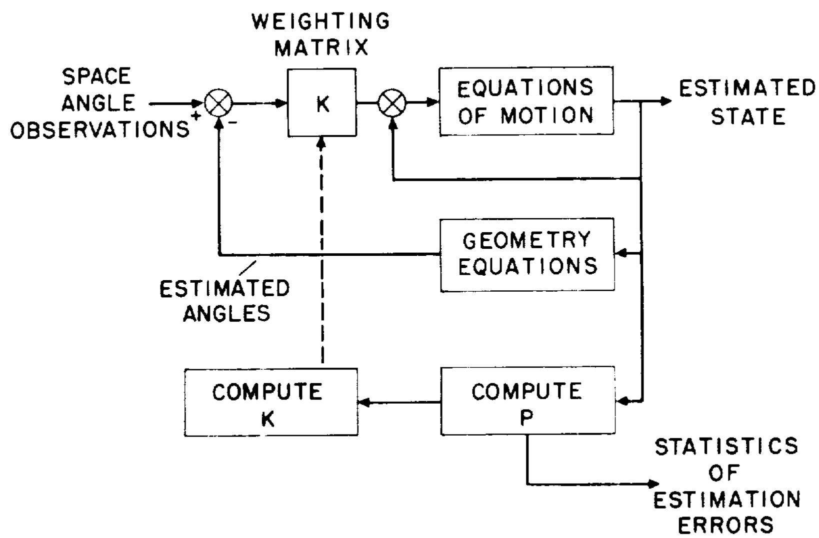
In proper perspective, this can be seen to represent in a general way a closed-loop control system as shown in block diagram form in sketch (b). The navigation and control aspects listed above comprise the forward loop of the system, and the applied control action (thrust) is fed back through the kinematics to influence the state which is the input to the forward loop of the system. In the present study we will assume that the control action will be intermittent and impulsive in nature (i.e., instantaneous velocity corrections). This assumption makes it possible to design the forward part of the loop independently of closed-loop considerations, nevertheless keeping in mind that the latter must eventually be considered. In reference 1 the design of the forward loop through the optimal filter was discussed. The remainder of the system and its closed-loop operation will be treated in this paper.



Sketch (b)

Theoretical Considerations

The optimal filter.- The theory of the optimal filter is given in detail in references 1 and 2 and will be discussed here only to the limited extent necessary to understand the results. The optimal trajectory determination scheme can be represented in the form illustrated in sketch (c). Observation inputs are assumed to consist of sets of angles measured at times t_k by sensors subject to random instrumentation errors. These errors are assumed to be statistically uncorrelated from one observation to the next.



Sketch (c)

The operation of the system is described as follows: After injection but before any observations have been made, the best estimate is based solely upon the best knowledge of injection conditions. Thus, the estimate of injection conditions is inserted as an initial condition on the equations of motion, which are integrated to keep the estimate of the state up to date. The gain K is zero except at times observations are made. When an observation is made at time t_k, the set of observed angles is compared with the estimated angles computed from the current estimate of the state variables. The difference is weighted by the matrix $K(t_k)$ to produce an incremental change in the estimated state at time t_k. The new estimated state variables then serve as new conditions on the equations of motion which are then integrated from time t_k until the time of the next observation, t_{k+1}, to maintain a continuous estimate of the state. As each observation is made, the process is repeated.

The weighting matrix $K(t_k)$ is seen to be the heart of the estimation procedure, and the objective is to develop the necessary equations from the theory whereby $K(t_k)$ can be computed. These equations are given in references 1 and 2. For solution they require a knowledge of the equations of motion, the relations between the observables and the state variables, and the statistics of injection errors and instrumentation errors. Involved as an intermediate step in the calculation of $K(t_k)$ is the computation of the covariance matrix of estimation errors, $P(t)$, which is a description of the statistics of the errors in the estimate and is therefore quite useful as a measure of the performance of the system. If the nature and the timing of observations are known a priori, then $K(t_k)$ and $P(t)$ can be computed before the flight and stored for use when needed. However, to allow for greater versatility these computations are envisioned as being performed in the spacecraft computer at the specific times they are needed.

The trajectory determination system, of course, operates as an integral part of the complete vehicle guidance and control system and must take into account the intermittent application of impulsive velocity corrections. When such corrective action is taken, the measured value of this action is introduced directly into the system as an instantaneous change in the estimate of the state, and the covariance matrix of the error in the measured value of the corrective action adds directly to P. Thereafter the system continues with its observation routine just as before with no loss of information due to the control action.

<u>The guidance equation</u>.- Linear prediction is used to calculate the velocity correction that must be applied for the vehicle to arrive at the desired end point. This method of prediction is based on the assumption that the actual trajectory can be accurately described in terms of linear perturbations around the reference trajectory. Given a set of perturbations in position and velocity at some time, t_o, the perturbations at a later time, t, will be a linear combination of the earlier ones. It is convenient to write this relationship in matrix form as follows:

$$x(t) = \Phi(t;t_o)x(t_o)$$

where the x's are 6×1 column matrices of perturbations in position and velocity and Φ is a 6×6 matrix, defined as the transition matrix between times t_o and t. The transition matrix between some earlier time, t, and the time of the reference end point is defined in this report as a prediction matrix and is designated by A. In equation form

$$x_E = Ax(t) \tag{1}$$

where x_E is a 6×1 matrix of position and velocity deviations at the end point. Provision has been made in the computer program to calculate both A and the transition matrices discussed in reference 1. The methods of computation are described in appendix B. Equation (1) may be written in partitioned form as

$$\begin{bmatrix} \bar{x}_E \\ \dot{\bar{x}}_E \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \dot{\bar{x}} \end{bmatrix} \tag{2}$$

where the $\bar{x}$ and $\dot{\bar{x}}$ are 3×1 matrices of position and velocity deviations, respectively, and the A_i are 3×3 matrices. It is desired to have the actual and reference trajectories coincide in position and velocity at the end point. In general, only three components of the vector x_E can be reduced to zero with a single velocity correction. For this reason the guidance equation is formulated so that only position errors will be made zero at the end point.

Equation (2) can be expanded to compute the end-point position error

$$\bar{x}_E = A_1\bar{x} + A_2\dot{\bar{x}} \tag{3}$$

If at time t a velocity correction, $\dot{\bar{x}}_G$, is added, the change in position at the end point will be

$$\bar{x}_E' = A_2\dot{\bar{x}}_G \tag{4}$$

For the position error at the end point to be reduced to zero

$$\bar{x}_E' + \bar{x}_E = 0 \tag{5}$$

or

$$A_1\bar{x} + A_2\dot{\bar{x}} + A_2\dot{\bar{x}}_G = 0 \tag{6}$$

which gives

$$\dot{\bar{x}}_G = -\left(A_2^{-1} A_1\right)\bar{x} - \dot{\bar{x}} \tag{7}$$

or

$$\dot{\bar{x}}_G = -\left(A_2^{-1} A_1 \quad I\right)x$$

where I is a 3×3 unit matrix.

For any given reference trajectory the submatrices, A_i, are prescribed functions of time and relate deviations from the reference trajectory at present time to those of a predetermined time of arrival at the end point. Therefore equation (7) implies guidance of the vehicle to a fixed position at a fixed time; in other words, the guidance equation represents a "fixed time of arrival system." This method of navigation may not be the best but the resulting guidance equation is simple and is adequate to demonstrate that the optimal filter combined with linear prediction can provide a satisfactory midcourse navigation system.

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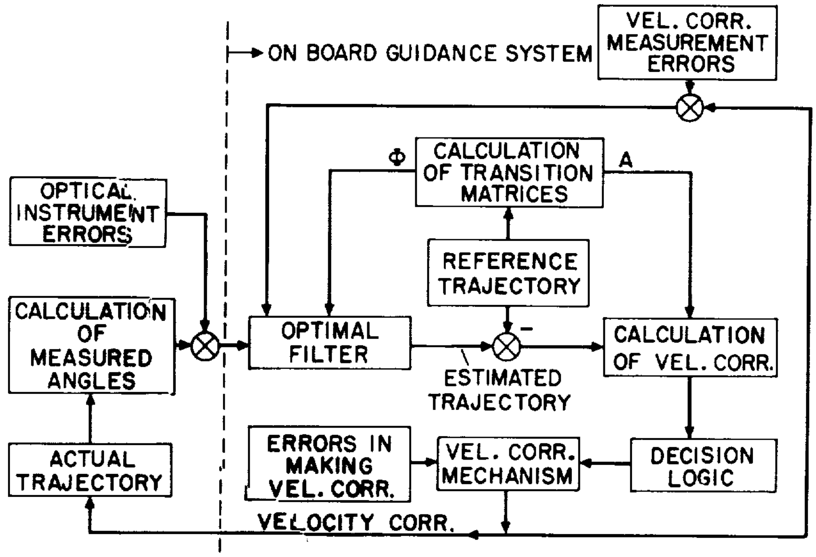
THE DIGITAL COMPUTER SIMULATION

Description of the System

The navigation system as simulated on a digital computer is illustrated in sketch (d). The portion to the left of the dotted line is not



Sketch (d)

part of the system aboard the vehicle. The blocks labeled "Actual Trajectory" and "Reference Trajectory" are solutions of the four-body nonlinear equations of motion in a geocentric nonrotating Cartesian coordinate system. Provision is made for perturbing the injection conditions of the actual trajectory in a random fashion from those of the reference. The equations of motion together with brief descriptions of the coordinate system and the trajectory computation are given in appendix A.

The optimal filter has been described previously. It will be regarded here simply as a device that gives the optimal estimate of position and velocity and the covariance matrix of the errors in those estimates.

The calculation of transition matrices, Φ and A, is a mechanization of the computations in appendix B referred to earlier. The calculation of velocity correction is a mechanization of the guidance equation, and the decision logic determines whether the indicated correction is actually to be made. Such a decision process in a practical system might be quite complex, but the logic used here simply calls for corrections at a few predetermined times.

When a velocity correction is applied, it is assumed that the vehicle is oriented by means of an attitude control system to the desired azimuth and elevation angles. A velocity control system is then energized to provide thrust until the desired velocity increment is achieved. These two systems are referred to as the velocity correction mechanism in sketch (d). In the simulation, random errors in correction magnitude and direction are generated and added to the desired velocity correction, to form the actual correction applied.

The block labeled "Vel. Corr. Measurement Errors" represents the error introduced by on-board measurement of the velocity correction (e.g., by integrating accelerometers on a stable platform). In the simulation the assumed errors in this measurement system must be added to the actual velocity correction to form the corrective velocity input to the optimal filter.

Assumptions of System Characteristics

The evaluation of the navigation system was made by considering only variations in those errors which could be readily accounted for statistically. The assumptions concerning the nature and magnitude of these statistical errors, are discussed in the following paragraphs.

Optical instrumentation.- First it was assumed that the vehicle will contain some sort of an inertial fixed reference system (e.g., a stable platform). For convenience of computing, these reference axes are taken as coinciding with those of the geocentric Cartesian system used for trajectory calculations. The calculated angles are the subtended half angle, γ, of the earth or moon and the elevation and azimuth angles, α and β, of the center line of the same body with respect to the reference axis system. The equations used for calculating these angles are given in appendix C. It was assumed that the optical instruments would measure the subtended angle and locate the center line of the planetary body either by manual observation of the disc or crescent or by automatic disc scanning. Such devices are subject to errors due to uncertainties in the observed surface such as atmospheric altitude fluctuations and nonspherical shape, and these errors become larger with decreasing range. The errors were assumed to be random with Gaussian distribution and zero mean. In addition, it was assumed that the errors in the various angles comprising one observation were independent of each other, had the same standard deviation, and were uncorrelated with the errors at other observation times. Since the angular errors become larger with decreasing range (i.e., with increasing magnitude of the subtended angle), the standard deviation of the errors in the observed angles, in seconds of arc, was assumed to be of the form

$$\sigma = \sqrt{K_1^2 + (K_2\gamma)^2}$$

where K_1 is the standard deviation of error in the basic optical system and K_2 reflects the range-dependent component.

<u>Sequence of observations.</u>- The sequence and spacing of observations were chosen with a view to what might be practical on such a flight. The on-board system was assumed to be capable of measuring a total of six angles (azimuth, elevation, and subtended angle of both earth and moon). The maximum amount of information theoretically available in one observation could be obtained by measuring these six angles simultaneously. However, the capability of sighting both bodies simultaneously would increase the complexity of the measuring instruments and probably complicate the vehicle design. For this reason it was decided that only one body at a time would be observed.

It appeared, from computer results not presented here, that in the immediate vicinity of the earth or moon it was best to observe only that body. On the other hand, more information could be obtained during most of the trajectory if the two bodies were observed alternately. It also seemed reasonable that more time must elapse between observations of two different bodies than between two sightings on the same one. For these reasons the following basic sequence of measurements was used. In the immediate vicinity of the moon or earth, as indicated in figure 1, only the nearer body was observed. For the remainder of the flight, first the earth was observed for 1/2 hour at 6-minute intervals (6 sets of observations), and then, after a 1/2-hour delay, the moon was observed for 1/2 hour. This procedure took place throughout most of the flight except for observationless periods near the points of entry into and exit from the moon's sphere of influence.[1]

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[1] The observationless periods were necessary to allow for translation of the origin of coordinates (see appendix A).

This sequence resulted in a total of 844 sets of observations for the complete trajectory.

<u>Velocity corrections</u>.- Six velocity corrections were made at times, measured from injection, of 0.5, 2, 3, 3.5, 5, and 6.45 days. These times are indicated on the reference trajectory in figure 1. The final correction was made at such a time as to allow approximately 1 hour of subsequent observation before re-entry. The information thus obtained would be useful for the terminal guidance system. Otherwise the number and spacing of corrections were chosen more or less arbitrarily. The velocity increments were applied in the middle of the half-hour delay periods between observations. This delay would allow time for orientation of the thrust axis and then reorientation of the vehicle for observation purposes.

<u>Injection errors</u>.- Injection errors in each of the Cartesian components of position and velocity were selected from a set of random numbers representing a Gaussian distribution.

<u>The standard case</u>.- The standard case was chosen as follows: Observation and velocity correction schedules were those described above. The values of K_1 and K_2 in the standard deviation of errors in the observed angles were chosen as 10 seconds of arc and 0.001, respectively, to give

$$\sigma = \sqrt{100 + (0.001\gamma)^2} \text{ seconds of arc}$$

The rms values of errors in making velocity corrections were 0.5° for each angle and 0.1 m/sec in magnitude of velocity. The rms error in measuring applied velocity corrections was taken as 1 cm/sec in each component. The rms values of injection errors were 1 km in each component of position and 1 m/sec in each component of velocity.

Calculation of Statistical Information

To conserve computer time several pertinent variables have been calculated in a statistical sense. For example, the covariance matrix, P, of errors in estimation discussed in reference 1 gives, in a single computer run, the mean square errors in estimation for a class or "ensemble" of trajectories; that is, it approximates the results that would be obtained from averages of all possible data runs with the following things in common:

1. Injection conditions with a given (Gaussian) statistical distribution around those of the reference trajectory;

2. The same set of angles observed at the same times;

3. The same method of calculation and times of application of velocity corrections; and

4. The same statistical distribution of errors in performing the above operations.

In addition to the covariance matrix of errors in estimation one would also wish to know R, the covariance matrix of differences between the actual and reference trajectories, $\tilde{p}_{rms}$, the rms error in prediction A
of position at the end point, $\tilde{u}_{rms}$, the rms velocity correction at each 5
point where a velocity correction is made, and the rms deviation in 5
radius of periapsis at the end point. The methods used for calculating 9
these quantities are outlined in appendix D.

RESULTS

All the results presented here use the reference trajectory shown in figure 1. This trajectory is entirely ballistic and lies approximately in the moon's orbital plane. Injection occurs at perigee with an altitude of 200 km at about 99.2 percent of escape velocity. The vehicle passes ahead of the moon and reaches perilune at a lunar altitude of 4766 km. The moon's gravitational attraction rotates the direction of flight so that the outgoing and return portions of the trajectory form the well-known figure-eight pattern. The vehicle's target on return is a vacuum perigee at an altitude located so the vehicle is traveling in approximately the same direction as the earth's rotation. The total flight time from injection to return perigee is 6.53 days. (This is the same reference trajectory as used in reference 1.)

The trajectory considered is generally suitable for the circumlunar mission because it has a relatively low energy, yet has a perilune not too large for effective lunar observations. Launch will almost certainly take place from Cape Canaveral and the reference trajectory used herein is not necessarily attainable from that launch site. Furthermore, there is no constraint imposed on the location of return perigee to provide for landing at a desired site. However, it is anticipated that the use of a different reference trajectory will not substantially alter the efficiency or general operating characteristics of the navigation system.

Errors Due to Linear Prediction

Before examining the operation of the navigation system it was considered desirable to obtain an estimate of the errors due to the linear prediction scheme used in the navigation system. Since the linear prediction matrices involved are obtained from perturbation equations of motion which are only approximately linear, errors at the end point must occur which are purely the result of this approximation. These errors can be evaluated by comparison with the solution of the complete nonlinear equations of motion for a system in which the determination of state (the position and velocity) and application of corrective velocity can be performed exactly. Because the linear perturbation equations are time variant, the evaluation must be made at several points along the trajectory. It is obviously impossible to consider every likely situation in such an evaluation, but a special case, which is of interest since it facilitates a quantitative evaluation of such errors, is outlined below.

Examination of the guidance equation (7), $\dot{\bar{x}}_G = -(A_2^{-1} A_1)\bar{x} - \dot{\bar{x}}$, shows that nonlinearity as a result of position deviation from the reference is the important consideration since velocity deviations are canceled by the application of a perfect velocity increment. The primary question to be answered is therefore how do the errors at the end point increase with increasing position deviation if equation (7) is used for computing velocity corrections?

In general,

$$X_{i_E} = f_i(X + x,\ t) \tag{8}$$

where X refers to components along the reference trajectory and x refers to deviations from the reference.

Expanding equation (8) in a Taylor series gives

$$X_{i_E} = f_i(X,t) + \sum_{j=1}^{6} \frac{\partial f_i}{\partial X_j} x_j + \frac{1}{2} \sum_{\substack{j=1\\k=1}}^{6} \frac{\partial^2 f_i}{\partial X_j\ \partial X_k} x_j x_k + \cdots \tag{9}$$

It can be seen that linear prediction theory accounts for the second term of equation (9). The errors in linear prediction are the higher order terms of the expansion. It can be reasonably well assumed that errors should be proportional to the first term neglected for small deviations from the reference trajectory. The error in linear prediction, δx_{i_E} is therefore

$$\delta x_{i_E} \approx \frac{1}{2} \sum_{\substack{j=1 \\ k=1}}^{6} \frac{\partial^2 f_i}{\partial X_j\, \partial X_k} x_j x_k = \frac{1}{2}\, x^T G_i x \tag{10}$$

where G_i is the 6×6 matrix of second partial derivatives. In partitioned form,

$$x^T G_i x = \left(\overline{x}^T\ \dot{\overline{x}}^T\right) \begin{bmatrix} G_{i_1} & G_{i_2} \\ G_{i_3} & G_{i_4} \end{bmatrix} \begin{bmatrix} \overline{x} \\ \dot{\overline{x}} \end{bmatrix} \tag{11}$$

$$= \overline{x}^T G_{i_1} \overline{x} + \dot{\overline{x}}^T G_{i_3} \overline{x} + \overline{x}^T G_{i_2} \dot{\overline{x}} + \dot{\overline{x}} G_{i_4} \dot{\overline{x}} \tag{12}$$

If at time t_1 a velocity correction is made by use of equation (7), then immediately after the velocity correction

$$\dot{\overline{x}}(t_1) = - A_2^{-1} A_1 \overline{x}(t_1) \tag{13}$$

Substitution of equation (13) in equation (12) with $x(t_1)$ defined as x_o yields

$$\begin{aligned} x_o^T G_i x_o = {} & \overline{x}_o^T G_{i_1} \overline{x}_o - \overline{x}_o^T \left(A_2^{-1} A_1\right)^T G_{i_3} \overline{x}_o - x_o^T G_{i_2} A_2^{-1} A_1 \overline{x}_o \\ & + \overline{x}_o^T \left(A_2^{-1} A_1\right)^T G_{i_4} A_2^{-1} A_1 \overline{x}_o \end{aligned} \tag{14}$$

If the magnitudes of the three components of $\overline{x}(t_1) = \overline{x}_o$ are chosen equal $\left(\text{i.e.,} \left|x_1(t_1)\right| = \left|x_2(t_1)\right| = \left|x_3(t_1)\right| = r_o/\sqrt{3}\right)$, then from equations (10) and (14)

$$\delta x_{i_E} \approx K_i r_o^2$$

and

$$r_E = \left(\sum_{i=} \delta x_{i_E}^2\right)^{1/2} = K r_o^2, \qquad \text{where} \qquad K = \left(K_1 + K_2 + K_3\right)^{1/2} \tag{15}$$

Thus we can conclude that to a first approximation the range error at the end point with linear prediction is proportional to the square of the deviation from the reference trajectory at the time of the velocity correction.

The validity of this line of reasoning was checked at four points on each phase (outbound and return) of the reference trajectory. Equal position deviations from the reference trajectory were made along each of the Cartesian coordinates by use of positive and negative values of 10, 100, and 1,000 kilometers in each component. Then for each of the six resulting cases the velocity was corrected according to equation (7) and the trajectories were computed to the end point by means of the complete nonlinear equations of motion. When the ratio of the magnitude of deviation at the end point to the square of the initial deviation was computed, it was found to be constant within a few percent for the two largest deviations. Disagreements in some cases with the smallest initial deviation could be attributed to loss of significant figures in making the velocity correction and computing the final errors. Initial displacements of 10,000 km were also used at a range of 150,000 km from the earth on both phases of the trajectory. The ratios of these cases corresponded quite closely to those for 100 and 1,000 km initial displacements.

A summary of the results of this study are shown in figure 2. The ratio of the magnitude of the end point error (inherent in the linear prediction scheme) to the square of the deviation at a particular range is plotted as a function of range from the center of the earth. The significance of these curves may be more easily understood by an example. In figure 3, to be discussed subsequently, rms position deviations are shown for an ensemble of trajectories. At 0.62 and 2.97 days on the outbound flight (corresponding to 150,000 and 350,000 kilometers range from the earth) the position deviations are 199 and 25 kilometers, respectively. When these numbers are multiplied by 3 to increase the probability that a single member of the ensemble will lie within these deviations and when the data of figure 2 are applied directly to these numbers (597 and 75 km), it is found that the error at perilune due to linear prediction is 2.07 and 0.026 kilometers, respectively.

If the data shown in figure 2 and the example cited are representative of likely cases, the following conclusions can be drawn. First, the inherent errors in the linear prediction scheme due to neglecting higher order terms are small for reasonable magnitudes of the deviation from the reference trajectory. Second, errors due to nonlinearity are greatest near the centers of attraction and reach a minimum at a range from the earth of about two-thirds of the earth-moon range.

Operation of the Navigation System

The purpose of this portion of the study was to determine whether the performance of the navigation system was satisfactory for reasonable magnitudes of those system errors which could be readily accounted for statistically; that is, random errors in observations, velocity corrections, and injection conditions were considered, but systematic errors, such as biases and imperfect knowledge of various astronomical constants, were left for subsequent studies. It was also desired to determine the effects of variations in the magnitudes of the various random errors and in the number and spacing of observations.

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Data for the evaluation of the navigation system were obtained with the aid of the digital computer for the following cases:

1. The standard case referred to earlier.

2. Same as case 1 except the errors in making and measuring velocity corrections were zero.

3. Same as case 1 except the rms value of observation errors was increased by a factor of 5, that is,

$$\sigma = \sqrt{2500 + (0.005\gamma)^2}$$

4. Same as case 1 except the rms value of injection errors was increased by a factor of 5 to 5 km and 5 m/sec.

5. Same as case 1 except the sequence of observations was so changed that one set of observations of the earth or the moon was made alternately at 2-hour intervals for a total of 77 observations.

Comparison of statistical data with one ensemble member.- Figure 3 shows, for the standard case, the rms position and velocity deviations between the actual and reference trajectories for an ensemble of runs. Also shown are the corresponding quantities for a specific run, that is, one member of the ensemble. Similar data were computed for all five of the cases mentioned and in all instances the correspondence of the individual and the ensemble data were well within what would be expected from theoretical considerations. Note that the velocity deviations increase rapidly near the centers of gravitational attraction (perilune time = 3.28 days, perigee time = 6.53 days). These increases are to be expected since the total velocity also increases rapidly. The increase in the rms position deviation prior to perigee time is a result of the increase in the velocity deviation. Since the gravitational center has a focusing effect on approach trajectories, it is reasonable to expect that this increase in position deviation occurs mainly along the path of the vehicle. This reasoning is borne out by comparison, to be given later, of position deviations at the time of reference perigee with those in perigee altitude.

Time histories of various statistical quantities of interest for all the five cases are presented in figures 4 through 8. The time histories involving trajectory estimation are given only to the time of the final observation. Figure 4 shows the rms errors in estimating the actual position and velocity of the vehicle at the time of that estimation. The curves are an average of the estimate just after each observation and do not depict what happens between observations.

Estimation errors.- A comparison of cases 1 and 2 shows the effects of errors in measuring velocity corrections. The effect is noticeable only during the return phase and is not significant there. As a result of the fact that the difference between the two cases is caused by errors in measurement of velocity corrections only, it is understandable that the effect is noticeable only when the error in estimating velocity is quite small. As can be seen from figure 4(d) (note the scale change from figure 4(c)), the error in estimating velocity achieves its minimum during the return phase. A comparison of cases 1 and 3 shows that increasing the errors in observation by a factor of 5 increases the error in estimate by a similar factor during most of the trajectory. This result should be expected in view of the fact that, except for a minor influence of injection errors and velocity correction measurement errors, the accuracy of the estimate of the trajectory is fundamentally dependent upon the errors of the observations. The fact that injection errors are relatively insignificant as far as errors in estimation are concerned can be seen by comparing cases 1 and 4. The only noticeable effect occurs during the first half day of the flight since after this time a sufficient number of observations has been made so that the knowledge of injection errors has little influence on the estimate of the trajectory.

From a comparison of cases 1, 3, and 5 it can be seen that the effect of reducing the number of observations is to a large degree equivalent to increasing the observation errors. Thus by reducing the errors in observation, one may make a lower number of observations and still retain as good an estimate.

Figures 4(c) and (d) show that the error in velocity estimate increases rapidly near the centers of gravitational attraction. The primary reason for this increase is that the velocity increases rapidly near the centers of attraction.

Prediction errors.- Figure 5 shows the rms error in prediction of the total end-point position deviations. This quantity represents the errors in estimation extended to the end point by linear prediction. The prediction equation is derived in appendix D. The calculation of this quantity was made under the assumption of perfect linearity; that is, prediction errors due to nonlinearities were not considered. In the absence of observations the prediction error remains constant. The very sharp decrease in prediction error during the early part of the flight is therefore a consequence of the observations.

On the outbound phase the rate of change of prediction error decreases with time and is quite small after about 1.5 days. As a consequence of the fact that it remains constant during observationless A
periods, it is likely that the rate of making observations could be 5
considerably decreased from this point to the moon without seriously 5
affecting perilune miss. Similarly on the return flight the observation 9
rate should be able to be considerably reduced during the period of 4.25 to 6.25 days without serious effect on the errors at entry.

The various parametric changes have an effect on prediction errors similar to that discussed previously for estimation errors.

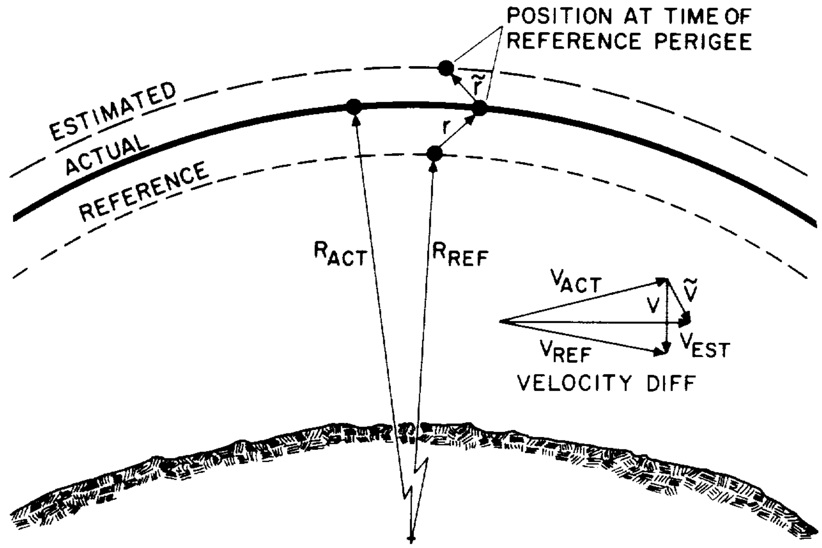
Indicated velocity correction.- Figure 6 shows the rms indicated velocity required to null the end point position error as a function of time. Since the reference trajectory and estimated trajectory (as determined by the optimal filter) are set equal to each other at injection, all the cases start with zero indicated correction. If no observations are taken, this quantity is zero throughout the flight. Since observations cause the estimated trajectory to converge on the actual trajectory, which in general differs from the estimate, the rms indicated velocity required increases with each observation (starting at injection). Prior to the first velocity correction, cases 1 and 2 obviously are identical since the observation sequences and injection errors are identical. For both cases 3 and 5, the indicated rms value of the first velocity correction is smaller than that in case 1, although this is not noticeable at the scale used in figure 6. This is a result of the fact that the estimated trajectory has not departed as far from the reference trajectory because less information is available from the observations. Thus, it might be said that the system automatically applies a confidence criterion, weighting the velocity corrections in accordance with the uncertainty in the estimate.

The discontinuities in the curves occur at the time of velocity corrections. Since the velocity corrections and the knowledge of position and velocity are imperfect, the rms correction required increases following a correction and new observations. For purposes of clarity only case 2 is shown in figure 6(b) after the final velocity correction. The behavior of the other curves is essentially the same and all become infinite at the time of reference perigee.

Deviations from the reference trajectory.- Figure 7 is a plot of the rms position deviation between actual and reference trajectories. The discontinuities in slope occur at the points where velocity corrections are made, as can be seen by comparison with figure 6. The divergence between cases 1 and 2 becomes relatively more important as return perigee is approached. This result would be expected because of the divergence in prediction errors and because the required velocity corrections become small compared to the errors in making them. In this connection it can be seen that the two cases with poor knowledge of the actual trajectory (3 and 5) show little or no improvement as the result of the final velocity correction. On the other hand, the standard case and the one with large injection errors (case 4) show a marked change and almost coincide at the final observation. A comparison of the curves of this figure with those of figure 4 shows that the improvement is greatest when the deviations from the reference are largest compared to the errors in knowledge of those deviations. In other words, the indicated velocity correction is proportional to the fraction of the end point error that can be predicted accurately.

The corresponding time histories for deviations in velocity are presented in figure 8. The step discontinuities occur at the times of the velocity corrections and their magnitudes indicate the magnitudes of the corrections made. Both in this figure and for the position deviations in figure 7, the case with no errors in making and measuring velocity corrections (case 2) does not differ significantly from the standard case until the return phase. This deviation is somewhat more pronounced than in the case of the estimation errors (fig. 4(d)) because the errors in making the correction, which are larger than those in measuring it, are involved here. Even the errors in applying the velocity correction, however, do not become significant until the errors of estimation become small. This fact would indicate that in the presence of an error in making the velocity correction which is independent of the magnitude of the correction,it would be better to time the later velocity corrections so that the magnitude of each one is larger.

Terminal conditions.- The most critical requirement of the midcourse guidance system is that it return the vehicle on a trajectory from which a safe re-entry can be made followed by landing at a predetermined site. Some quantities which show how well this end is accomplished are shown in sketch (e). The distance, r, is the magnitude of the vehicle's position deviation from the reference trajectory. The distance, $\tilde{r}$, is the magnitude of error in estimating the vehicle's position, while v and $\tilde{v}$ are the corresponding deviations in velocity.



Sketch (e)

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These quantities are essentially the same at the time of reference perigee as they would be at re-entry into the earth's atmosphere. The quantities $\tilde{r}$ and $\tilde{v}$ at the time of reference perigee indicate the accuracy of information supplied to the terminal guidance system. The perigee of the actual trajectory can be expected to occur at a different time than that of the reference, and although the deviation in radius of perigee determines the possibility of making a safe re-entry, it does not necessarily satisfy the requirement for point landing.

The rms values of these quantities and others of interest were calculated for the different error assumptions by linear statistical methods and are presented in table I. In the first row are listed results for the standard case referred to previously. Note that the rms variation in perigee height is only 0.6 km, indicating a highly successful survival potential for the spacecraft and its occupants. The next two numbers, 11.5 km and 11.0 m/sec for the total rms range and velocity, respectively, are given at vacuum perigee but are of the same order of magnitude at atmosphere entry. Since the spacecraft is re-entering at near parabolic velocity, the entry flight-path angle variation is less than 0.001 radian and the error in range can easily be eliminated during terminal guidance. A second set of data, those of error in knowledge of position and velocity, are given at the time of reference perigee as 7.7 km and 6.7 m/sec, respectively. These latter quantities influence the terminal guidance system, but the resultant miss on landing was not calculated.

A conservative estimate of the total root-mean-square corrective velocity required for the ensemble of trajectories during the 6-1/2-day flight is obtained by adding the root-mean-square values for the individual correction times. The total corrective velocity for the standard case has a value of 15.3 meters per second. It is desirable to have some estimate of the percentage of the vehicle's weight which must be allocated to fuel for the midcourse guidance corrections. For this purpose the weight percentages were calculated using a specific impulse of 300 seconds and the total rms velocity correction required. These results are listed in the last column of table I. Several times this amount of fuel will be needed for confidence in the safety of the mission, but the requirement is still quite modest.

The effects of various parametric changes from the standard are shown in the next four rows. For example, if one could make and measure velocity corrections perfectly, then the perigee errors are reduced but the reduction in total corrective velocity is comparatively small.

Increasing the errors in observations by a factor of 5 increases all the terminal errors by a factor of 2 to 4 but velocity increases only by about 50 percent. As might be expected, since the trajectory determination system is quite accurate, increasing injection errors by a factor of 5 increases the total velocity correction required by somewhat less than 5 but has little effect on terminal errors. A comparison of the data in the third and fifth rows shows that a good measurement system allows the liberty of taking fewer observations to achieve the same accuracy at perigee. Even though the total number of observations was decreased from 844 to 77, the terminal errors were only approximately doubled and the increase in corrective velocity was small.

The equivalent data for perilune are presente[illegible]
metric changes other than increased injectio[illegible]
on the total corrective velocity for this
differences between r and the deviation in [illegible]
as pronounced as in the case of return perigee, bu[illegible]
large.

The rms position errors, r, are very nearly equal to the prediction errors at the time of the final velocity correction. Delay of the final correction would cause r to approach $\tilde{r}$ as a lower limit, but the rapid increase in corrective velocity required would make any significant reduction in r impractical. On the other hand, the amount of fuel necessary could quite probably be reduced if a larger position deviation at periapsis could be tolerated.

The rms deviation in radius of periapsis $R_{ref} - R_{act}$, particularly at the earth, is much smaller than the corresponding rms position deviation from the reference trajectory at the time of reference periapsis.

This fact indicates an added safety factor for emergency conditions since the probability of a safe return to an undetermined landing site is considerably higher than that of landing at a particular site. Further studies are needed to determine the effects of deviations from the reference trajectory at the time of perigee on the accuracy of attaining the landing site. However, these data indicate that the final velocity correction should be based on the characteristics of the terminal portion of the trajectory.

Increasing the rms value of injection errors had negligible effect on the end point position errors. The data show, on the other hand, that the total rms velocity correction required is almost a constant multiple of rms injection errors.

The data in the tables show the system to be efficient and accurate. However, it must be borne in mind that systematic errors were not considered in obtaining these results. Likewise no attempt was made to make the number and spacing of either observations or velocity corrections optimum.

Computer requirements.- An additional item of information which may be of interest is the computer requirements. The simulation uses 13,233 words of storage and requires about 3 hours (including set-up time) on the IBM 704. A Fortran program was written which deleted all parts of the simulation, such as integration of the actual trajectory, not necessary to the on-board navigation system. This revision reduced storage requirements by about a factor of 2. Additional simplifications can probably be made to further reduce the storage requirements. Data on the time requirements of the simplified program are not available, [illegible] should be a substantial reduction.

[illegible]CLUDING REMARKS

The midcourse navigation system using the optimal filter and linear prediction performs satisfactorily in the presence of the errors assumed. With proper scheduling the number of observations can be kept within reason and the accuracy required of these observations should be within the state of the art by the time the mission is attempted. In addition the amount of fuel necessary for midcourse guidance is not excessive and extreme accuracy in applying the velocity corrections is not required.

It is true that systematic errors such as bias and imperfect knowledge of astronomical constants have not been considered in this study, and also no attempt has been made to determine the optimum number, type, and spacing of observations or the times of making velocity corrections. Work is now under way to evaluate the effects of systematic errors. The distribution of injection errors assumed (equal in each Cartesian coordinate) is incorrect; however, it is not anticipated that use of the correct distribution when available will have an appreciable effect on the results except for the fuel requirement, which will be larger for larger injection errors.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Dec. 4, 1961

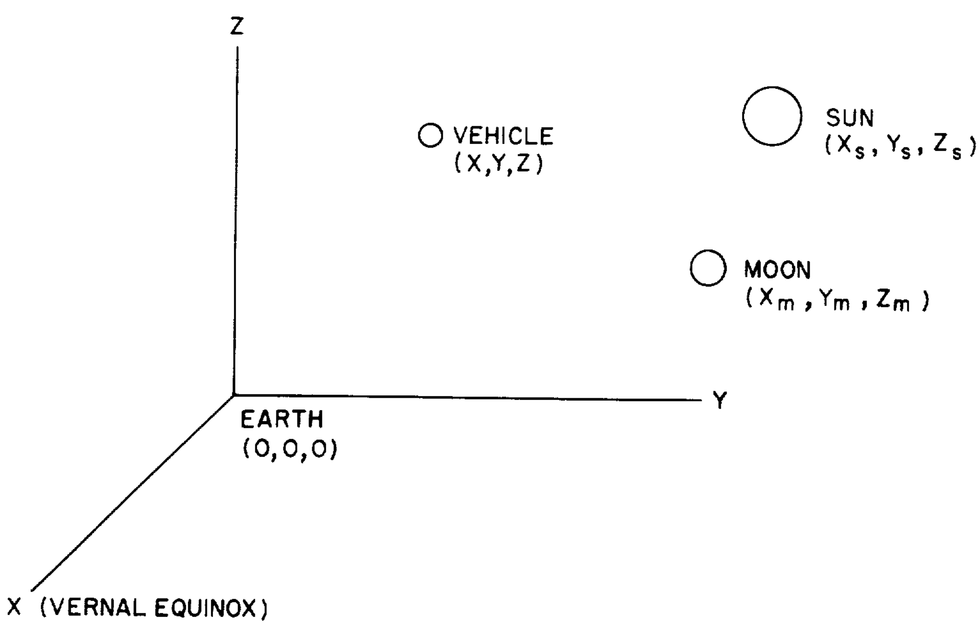
APPENDIX A

NONLINEAR EQUATIONS OF MOTION FOR TRAJECTORY CALCULATIONS

The equations of motion are derived under the following assumptions:

1. A restricted four-body system is sufficiently accurate.
2. The second harmonic term of the earth's oblateness is sufficient.
3. The sun and moon are spherical and homogeneous.

The coordinate system is Cartesian and geocentric. The Z axis lies along the earth's polar axis, positive to the north. The X and Y axes lie in the equatorial plane with the positive X axis in the direction of the vernal equinox. The Y axis is oriented so as to form the right handed orthogonal system shown in sketch (f).



Sketch (f)

The equations of motion are derived by methods given in reference 3. They are as follows:

$$\ddot{X} = \frac{-\mu_e X}{R_e^3}\left[1 + J\left(\frac{a}{R_e}\right)^2\left(1 - 5\,\frac{Z^2}{R_e^2}\right)\right]$$

$$- \frac{\mu_m}{\Delta_m^3}(X - X_m) - \frac{\mu_m X_m}{R_m^3} - \frac{\mu_s (X - X_s)}{\Delta_s^3} - \frac{\mu_s X_s}{R_s^3} \qquad \text{(A1)}$$

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$$\ddot{Y} = \frac{-\mu_e Y}{R_e^3}\left[1 + J\left(\frac{a}{R}\right)^2\left(1 - 5\,\frac{Z^2}{R_e^2}\right)\right]$$

$$- \frac{\mu_m}{\Delta_m^3}(Y - Y_m) - \frac{\mu_m Y_m}{R_m^3} - \frac{\mu_s (Y - Y_s)}{\Delta_s^3} - \frac{\mu_s Z_s}{R_s^3} \qquad \text{(A2)}$$

$$\ddot{Z} = \frac{-\mu_e Z}{R_e^3}\left[1 + J\left(\frac{a}{R_e}\right)^2\left(3 - \frac{5Z^2}{R_e^2}\right)\right]$$

$$- \frac{\mu_m (Z - Z_m)}{\Delta_m^3} - \frac{\mu_m Z_m}{R_m^3} - \frac{\mu_s (Z - Z_s)}{\Delta_s^3} - \frac{\mu_s Z_s}{R_s^3} \qquad \text{(A3)}$$

where

$$R_e = \sqrt{X^2 + Y^2 + Z^2}$$

$$R_m = \sqrt{X_m^2 + Y_m^2 + Z_m^2}$$

$$R_s = \sqrt{X_s^2 + Y_s^2 + Z_s^2}$$

$$\Delta_m = \sqrt{(X - X_m)^2 + (Y - Y_m)^2 + (Z - Z_m)^2}$$

$$\Delta_s = \sqrt{(X - X_s)^2 + (Y - Y_s)^2 + (Z - Z_s)^2}$$

$$\mu_e = 3.986135 \times 10^{14} \quad m^3/sec^2$$

$$\mu_m = 4.89820 \times 10^{12} \quad m^3/sec^2$$

$$\mu_s = 1.3253 \times 10^{20} \quad m^3/sec^2$$

$$a = \text{equatorial radius of the earth} = 6.37826 \times 10^6 \quad m$$

$$J = 1.6246 \times 10^{-3}$$

The terms involving only distances between the earth and moon or earth and sun, such as $\mu_m X_m / R_m^3$, arise from the fact that the coordinate system is not inertial. These terms account for the accelerations of the coordinate system with respect to inertial space.

The equations of motion for the vehicle are solved by means of a Cowell "second-sum" method. A fourth order Runge-Kutta method is used to start the integration and to change the step size during the flight. The positions of the sun and moon are obtained by interpolation of data from magnetic tape ephemerides. Within the sphere of influence of the moon, a lunar radius of 66,000 km, the origin of coordinates is translated to the center of the moon. Since no rotation is performed, the definitions of perturbations from the reference (see appendix B) remain the same.

APPENDIX B

CALCULATION OF TRANSITION AND PREDICTION MATRICES

The transition matrices used in the navigation system are obtained by solving linear differential equations that represent perturbations of the actual trajectory from the reference. These perturbation differential equations are derived as follows:

The nonlinear equations of motion given in appendix A can be written in the form

$$\left.\begin{aligned} \ddot{X} &= F_1(X, Y, Z, t) \\ \ddot{Y} &= F_2(X, Y, Z, t) \\ \ddot{Z} &= F_3(X, Y, Z, t) \end{aligned}\right\} \tag{B1}$$

It is desired to find linear differential equations for small deviations from the reference. These equations may be found by expanding equations (B1) about the reference trajectory in a Taylor series and dropping all terms except the first order.

$$\left.\begin{aligned} \delta\ddot{X} &= \frac{\partial F_1}{\partial X}\,\delta X + \frac{\partial F_1}{\partial Y}\,\delta Y + \frac{\partial F_1}{\partial Z}\,\delta Z \\ \delta\ddot{Y} &= \frac{\partial F_2}{\partial X}\,\delta X + \frac{\partial F_2}{\partial Y}\,\delta Y + \frac{\partial F_2}{\partial Z}\,\delta Z \\ \delta\ddot{Z} &= \frac{\partial F_3}{\partial X}\,\delta X + \frac{\partial F_3}{\partial Y}\,\delta Y + \frac{\partial F_3}{\partial Z}\,\delta Z \end{aligned}\right\} \tag{B2}$$

It is convenient to deal with systems of linear differential equations in multiple variables in matrix form. For this purpose it is generally desirable to reduce the system to a set of first-order equations as follows.

Define

$$\left.\begin{aligned} x_1 &= \delta X \qquad & x_4 &= \delta\dot{X} \\ x_2 &= \delta Y \qquad & x_5 &= \delta\dot{Y} \\ x_3 &= \delta Z \qquad & x_6 &= \delta\dot{Z} \end{aligned}\right\} \tag{B3}$$

Then the system of perturbation equations can be written in matrix form as

$$\frac{dx}{dt} = F(t)x \tag{B4}$$

where F is a 6×6 matrix of coefficients and x is a 6×1 column vector of the x_i defined above. From equations (B2) and (B3), equation (B4) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial Z} & 0 & 0 & 0 \\ \frac{\partial F_2}{\partial X} & \frac{\partial F_2}{\partial Y} & \frac{\partial F_2}{\partial Z} & 0 & 0 & 0 \\ \frac{\partial F_3}{\partial X} & \frac{\partial F_3}{\partial Y} & \frac{\partial F_3}{\partial Z} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \tag{B5}$$

Consider any system of homogeneous linear first-order differential equations written in matrix form

$$\frac{dx}{dt} = F(t)x \tag{B6}$$

where F is an $n \times n$ matrix of time variant coefficients and x is an $n \times 1$ column vector of dependent variables. It is shown in reference 4 that if U is a nonsingular matrix having n columns of n linearly independent solutions of (B6), then U (defined as a fundamental matrix) is a solution of

$$\frac{dU}{dt} = F(t)U \tag{B7}$$

where $U(t_o)$ is a constant matrix. As a special case Φ is defined as being the U obtained when $U(t_o)$ is the unit matrix. Thus Φ can be obtained one column at a time if equation (B6) is solved n times, each with a different member of $x(t_o)$ set equal to unity and all the other members set equal to zero. Once Φ is obtained the solution of $x(t)$ for any given set of initial conditions x_o is given by

$$x(t) = \Phi(t)x_o \tag{B8}$$

The matrix $\Phi(t)$ represents the "transition" in the states of the system of equations between time t_o and t and may be written as $\Phi(t;t_o)$ to indicate this fact. If equation (B5) is solved in this manner, the resulting transition matrix relates deviations from the reference trajectory at time t to the initial deviations at time t_o. The transition matrix $\Phi(t_2;t_1)$ between any two times on the reference trajectory may be calculated in the same manner as $\Phi(t;t_o)$.

The calculation is performed in the IBM 704 simulation by solving six sets of perturbation equations, each with a unit initial condition on one of the x_i, between succeeding times of observations. After each observation the initial conditions are reset, to unity or zero, and the computation is carried out until the next observation. This procedure was found to have certain practical advantages (discussed in ref. 1) which might also apply to the computer on board the spaceship.

The prediction matrix, $A(t_E;t_o)$, from the initial time to the time at the end point t_E is precalculated and stored in the computer. The end point state can be calculated as

$$x(t_E) = A(t_E;t_o)x(t_o) \tag{B9}$$

Then if the transition matrix from time t_o to some intermediate time t_k is known

$$x(t_k) = \Phi(t_k,t_o)x(t_o) \tag{B10}$$

Combining (B9) and (B10) gives

$$x(t_E) = A(t_E;t_o)\Phi^{-1}(t_k,t_o)x(t_k) \tag{B11}$$

So the prediction matrix relating deviations at the end time to those at some earlier time is

$$A(t_E;t_k) = A(t_E;t_o)\Phi^{-1}(t_k;t_o)$$

or, in general,

$$A(t_E;t_k) = A(t_E;t_o)\prod_{i=1}^{k}\Phi^{-1}(t_i;t_{i-1})$$

The $\Phi(t_i;t_{i-1})$ are computed for use in the optimal filter and it will be shown that they can be inverted by simply rearranging terms with some associated sign changes. Hence, by the use of a single stored matrix and a few matrix operations it is possible to compute the prediction matrix at each observation time. It should be noted that when the transition matrices are computed with the use of the estimated trajectory, as was done in reference 1, errors will build up in the prediction matrix due to the differences between estimated and reference trajectory. Such errors could be reduced if the calculated prediction matrix were replaced with the correct one stored at a few predetermined times along the reference trajectory.

The inversion property mentioned above is derived here. If equation (B8) is premultiplied by $\Phi^{-1}(t)$

$$x_o = \Phi^{-1}(t)x(t) \tag{B12}$$

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It can be seen from equation (B12) that $\Phi^{-1}(t)$ relates the set of deviations at time t_o which will produce unit deviations at time t. For the navigation problem it is desired to have the matrix which relates unit deviations at time t to the deviations at some end time t_E; that is, if $\Phi(t)$ were computed backward in time from t_E, its inverse would be the desired prediction matrix.

Now consider the adjoint system of equations defined by

$$\frac{d\lambda}{dt} = -F^T \lambda \tag{B13}$$

This system has a fundamental matrix Λ obtained in the same fashion as Φ, and it is shown in reference 4 that

$$\Lambda(t) = \Phi(t)^{T^{-1}}$$

or

$$\Phi^{-1}(t) = \Lambda^T(t) \tag{B14}$$

Hence the desired prediction matrix can be found when the adjoint system is solved backward in time and the resulting transition matrix is transposed. (This prediction matrix could then be stored at discrete times and velocity corrections for intermediate times could be found by interpolation.)

Equation (B5) can be written in partitioned form as

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ \bar{F} & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \tag{B15}$$

The definitions of $\bar{x}_1$, $\bar{x}_2$, and the submatrices of F can be seen from comparison of equation (B15) with (B5). Expanding (B15) gives

$$\dot{\bar{x}}_1 = \bar{x}_2 \tag{B16}$$

$$\dot{\bar{x}}_2 = \overline{F}\bar{x}_1 \tag{B17}$$

or combining (B16) and (B17) gives

$$\ddot{\bar{x}}_1 = \overline{F}\bar{x}_1 \tag{B18}$$

(Equation (B18) is identical with (B2) and is derived in this fashion only to show correspondence with the adjoint system.) From equations (B13) and (B15)

$$\begin{bmatrix} \dot{\bar{\lambda}}_1 \\ \dot{\bar{\lambda}}_2 \end{bmatrix} = - \begin{bmatrix} 0 & \overline{F}^T \\ I & 0 \end{bmatrix} \begin{bmatrix} \bar{\lambda}_1 \\ \bar{\lambda}_2 \end{bmatrix} \tag{B19}$$

where the $\bar{\lambda}_i$ are defined in the same fashion as the $\bar{x}_i$. Expanding equation (B19) gives

$$\dot{\bar{\lambda}}_1 = - \overline{F}^T \bar{\lambda}_2 \tag{B20}$$

$$\dot{\bar{\lambda}}_2 = -\bar{\lambda}_1 \tag{B21}$$

or combining (B20) and (B21) gives

$$\ddot{\bar{\lambda}}_2 = \overline{F}^T \bar{\lambda}_2 \tag{B22}$$

The matrix $\overline{F}$, however, is symmetrical[1] so that

$$\ddot{\bar{\lambda}}_2 = \overline{F\lambda}_2 \tag{B23}$$

Thus equations (B18) and (B23) are identical in form, while equations (B16) and (B21) differ only in sign.

Assume that $x_1(t_0)$ is set equal to unity, the other initial conditions being zero, and equation (B18) is solved for x_1, x_2, x_3, and their first derivatives. By letting $x_4 = \dot{x}_1$, $x_5 = \dot{x}_2$, and $x_6 = \dot{x}_3$ one

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[1]This property of symmetry is a result of the fact that the F_i of the equations of motion (B1) are the first partial derivatives of the gravitational potential and are continuous in the region of interest; that is,

$$F_1 = \frac{\partial\psi}{\partial X}$$

$$F_2 = \frac{\partial\psi}{\partial Y}$$

$$F_3 = \frac{\partial\psi}{\partial Z}$$

where ψ is the gravitational potential. Equation (B2) could therefore be written as

$$\begin{pmatrix} \delta\ddot{X} \\ \delta\ddot{Y} \\ \delta\ddot{Z} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2\psi}{\partial X^2} & \frac{\partial^2\psi}{\partial Y\,\partial X} & \frac{\partial^2\psi}{\partial X\,\partial Z} \\ \frac{\partial^2\psi}{\partial X\,\partial Y} & \frac{\partial^2\psi}{\partial Y^2} & \frac{\partial^2\psi}{\partial Z\,\partial Y} \\ \frac{\partial^2\psi}{\partial X\,\partial Z} & \frac{\partial^2\psi}{\partial Y\,\partial Z} & \frac{\partial^2\psi}{\partial Z^2} \end{pmatrix} \begin{pmatrix} \delta X \\ \delta Y \\ \delta Z \end{pmatrix}$$

Since $\frac{\partial^2\psi}{\partial Y\,\partial X} = \frac{\partial^2\psi}{\partial X\,\partial Y}$, etc., $F = F^T$

obtains the first column of the Φ matrix. However, in solving equation (B18) with a unit initial condition on x_1 one also solves equation (B23) with a unit initial condition on λ_4, and by letting $\dot{\lambda}_1 = -\lambda_4$, $\dot{\lambda}_2 = -\lambda_5$, and $\dot{\lambda}_3 = -\lambda_6$ one obtains the fourth column in the Λ matrix. Similarly, placing a positive unit initial condition on $\dot{x}_1$, $\dot{x}_2$, or $\dot{x}_3$ (i.e., on x_4, x_5, or x_6) is equivalent to putting a negative unit initial condition on λ_1, λ_2, or λ_3. If this analysis is carried through completely, it is found that if Λ and Φ are written in partitioned form as

$$\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{bmatrix} \quad \text{and} \quad \Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix}$$

then

$$\Lambda = \begin{bmatrix} \Phi_4 & -\Phi_3 \\ -\Phi_2 & \Phi_1 \end{bmatrix} \tag{B24}$$

but from equation (B14)

$$\Phi^{-1} = \Lambda^T$$

therefore

$$\Phi^{-1} = \begin{bmatrix} \Phi_4{}^T & -\Phi_2{}^T \\ -\Phi_3{}^T & \Phi_1{}^T \end{bmatrix} \tag{B25}$$

and similarly

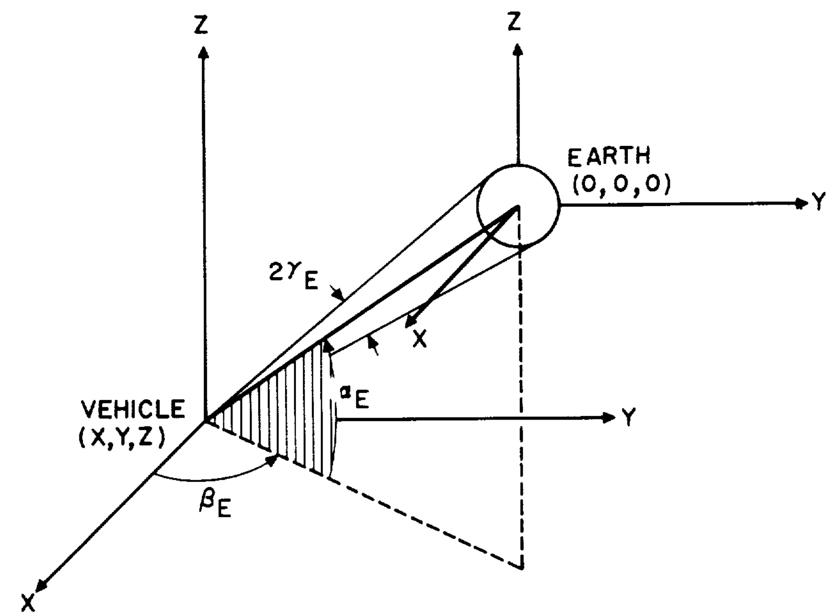
$$\Lambda^{-1} = \begin{bmatrix} \Lambda_4{}^T & -\Lambda_2{}^T \\ -\Lambda_3{}^T & \Lambda_1{}^T \end{bmatrix} \tag{B26}$$

The above analysis applies equally well to the trajectories run backward in time and for any initial and final times. Thus it is seen that because of the symmetrical properties of the perturbation equations, the transition matrix relating any two points on the trajectory can be inverted in this manner. In fact, any set of linear first-order differential equations which can be rewritten as a set of even order (i.e., second, fourth, etc.) equations with a symmetrical coefficient matrix will exhibit this property.

APPENDIX C

CALCULATION OF MEASURED ANGLES

As was mentioned in the description of the simulated system, it is assumed that the vehicle contains a reference system alined with the Cartesian coordinate system described in appendix A. The geometry of the situation is illustrated in sketch (g). The direction of the line A559



Sketch (g)

of sight is specified by the elevation angle α_e and the azimuth angle β_e. The subtended earth angle is $2\gamma_e$. The angle β_e is assumed to be measured counterclockwise from the vernal equinox (the X axis) and α_e is taken to be positive if the vehicle is below the equatorial plane.

The equations that relate the angles α_e, β_e, and γ_e to vehicle positions can be derived from sketch (g). They are:

$$\alpha_e = -\sin^{-1}\left(\frac{Z}{R_e}\right)$$

$$\beta_e = \sin^{-1} \frac{-Y}{\sqrt{X^2 + Y^2}} = \cos^{-1} \frac{-X}{\sqrt{X^2 + Y^2}}$$

$$\gamma_e = \sin^{-1} \frac{a}{R_e}$$

where a is the equatorial radius of the earth and

$$R_e = \sqrt{X^2 + Y^2 + Z^2}$$

A similar set of equations can be derived for the case when the moon is observed. With the notation of appendix A, these equations are:

$$\alpha_m = \sin^{-1} \left(\frac{Z_m - Z}{\Delta_m}\right)$$

$$\beta_m = \tan^{-1} \left(\frac{Y_m - Y}{X_m - X}\right)$$

$$\gamma_m = \sin^{-1} \left(\frac{a_m}{\Delta_m}\right)$$

where a_m is the radius of the moon, X_m, Y_m, and Z_m are the coordinates of the moon's position and

$$\Delta_m = \sqrt{(X - X_m)^2 + (Y - Y_m)^2 + (Z - Z_m)^2}$$

APPENDIX D

COMPUTATION OF STATISTICAL INFORMATION

It was stated in the text that statistical quantities of interest could be computed for the entire ensemble of trajectories having certain things in common in one run on the digital computer. From this statistical information one can determine the probability, or confidence factor, of a randomly chosen member of the ensemble meeting desired conditions. The matrix equations used to compute most of this information are derived below. It can be seen from the derivations that the accuracy with which these statistical quantities represent the entire ensemble depends on the validity of the linear perturbation equations discussed in appendix B.

The matrix R is defined as being the covariance matrix of deviations between the actual and reference trajectories. At injection, R is the covariance matrix of injection errors and is assumed to be known. By definition for any value of time

$$R = E(xx^T) \tag{D1}$$

where x is the deviation between actual and reference trajectories. If $x(t_o)$ is known then

$$x(t_1) = \Phi(t_1;t_o)\ x(t_o) \tag{D2}$$

where $\Phi(t_1;t_o)$ is the transition matrix between t_1 and t_o (see appendix B). By substitution of equation (D2) into (D1)

$$R(t_1) = E\left[\Phi(t_1;t_o)\ x(t_o)\ x^T(t_o)\ \Phi(t_1;t_o)^T\right] \tag{D3}$$

but $\Phi(t_1;t_o)$ is a constant matrix and may be removed from the brackets. As a result,

$$\begin{aligned} R(t_1) &= \Phi(t_1;t_o)\ E\left[x(t_o)\ x^T(t_o)\right]\ \Phi(t_1;t_o)^T \\ &= \Phi(t_1;t_o)\ R(t_o)\ \Phi(t_1;t_o)^T \end{aligned}$$

or, in general,

$$R(t_k) = \Phi(t_k;t_{k-1})R(t_{k-1})\Phi^T(t_k;t_{k-1}) \tag{D4}$$

Equation (D4) is valid provided no velocity correction is made in the time interval under consideration. The change in R due to a velocity correction must be found if R is to be determined over the entire trajectory.

The covariance matrix of errors in estimating the actual trajectory, designated P, is identical with R before the first observation is made. The computation of changes in P due to observations or velocity corrections is presented in reference 1 and will not be reproduced here. A knowledge of P is necessary for computing the effects of a velocity correction on R and for that purpose it is assumed that P is known.

After a velocity correction the state vector, x_c, of deviations between the actual and reference trajectories is

$$x_c = x + x_G + x_Q \tag{D5}$$

Here x_G is a 6×1 column vector whose three position terms are zero and whose velocity terms are the components of $\bar{x}_G$ calculated in equation (7). Similarly x_Q is a 6×1 vector having zeros in the first three terms and the components of a 3×1 vector, $\bar{x}_Q$, of errors in making the velocity correction for the others. This notation can be used to rewrite equation (7) as

$$x_G = - \begin{bmatrix} 0 & 0 \\ A_2^{-1} A_1 & I \end{bmatrix} \hat{x} \tag{D6}$$

The zeros are 3×3 null matrices, I is a 3×3 unit matrix, and $\hat{x}$ is the 6×1 column vector of estimated deviations from the reference.

For convenience in writing let

$$G = - \begin{bmatrix} 0 & 0 \\ A_2^{-1} A_1 & I \end{bmatrix} \tag{D7}$$

After a velocity correction

$$R = E(x_c x_c^T)$$

but

$$E(x_c x_c^T) = E\left[(x + G\hat{x} + x_Q)\ (x^T + \hat{x}^T G^T + x_Q^T)\right] \tag{D8}$$

Since x_Q is assumed to be uncorrelated with x or for different times, $E(\hat{x}x_Q{}^T) = E(xx_Q{}^T) = 0$. With this fact together with the identity that for any vector, y, $E(yy^T) = E(y^Ty)$ equation (D8) can be reduced to

$$E(x_cx_c{}^T) = R + E(x\hat{x}^TG^T + G\hat{x}x^T + G\hat{x}\hat{x}^TG^T) + S, \tag{D9}$$

where

$$S = E(x_Qx_Q{}^T) \tag{D10}$$

It is shown in reference 2 that when $\hat{x}$ is defined by the expression $\hat{x} = x + \tilde{x}$, $E(\hat{x}\tilde{x}^T) = 0$. Thus,

$$E(\hat{x}x^T) = E(\hat{x}\hat{x}^T)$$

Substituting $\hat{x} = x - \tilde{x}$ into the left side of the above equation gives

$$E(x\hat{x}^T) = E(xx^T - x\tilde{x}^T)$$

but

$$E(x\tilde{x}^T) = E(\hat{x}\tilde{x}^T + \tilde{x}\tilde{x}^T) = E(\tilde{x}\tilde{x}^T)$$

so that

$$E(x\hat{x}^T) = E(\hat{x}\hat{x}^T) = R-P$$

where P is defined (as in ref. 1) as

$$P = E(\tilde{x}\tilde{x}^T)$$

Substituting into equation (D9) and collecting terms gives

$$E(x_cx_c{}^T) = (I + G)(R-P)(I + G)^T + P + S \tag{D11}$$

Equations (D4) and (D10) can be used to compute R for the entire trajectory provided the covariance matrix, S, of errors in making velocity corrections is known. The matrix S is computed on the basis of the assumed velocity correction system described in the text. For simplicity it is assumed that the reference system on board the vehicle is alined with the geocentric Cartesian coordinate system used for the equations of motion and described in appendix A. The azimuth angle, ψ, and elevation angle, θ, of the thrust axis and the magnitude, u, of the velocity increment are measured as described in the text. It is desired to compute a matrix, C, which transforms the errors in mechanizing ψ, θ, and u into errors in the Cartesian components of velocity. The transformation takes the form

$$\begin{bmatrix} \dot{x}_Q \\ \dot{y}_Q \\ \dot{z}_Q \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \delta\theta \\ \delta\psi \\ \delta u \end{bmatrix}$$

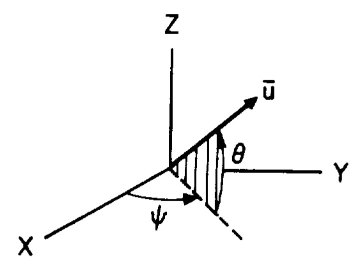
or

$$\dot{\bar{x}}_Q = C\bar{\theta} \tag{D12}$$

The components $\dot{x}_a$, $\dot{y}_a$, and $\dot{z}_a$ of the velocity correction actually applied are

$$\left.\begin{aligned} \dot{x}_a &= u \cos\theta \cos\psi \\ \dot{y}_a &= u \cos\theta \sin\psi \\ \dot{z}_a &= u \sin\theta \end{aligned}\right\} \tag{D13}$$

Equations (D13) are readily derived from sketch (h): The components of the matrix, C, are found when the partial derivatives of equations (D13) with respect to θ, ψ, and u are taken.



Sketch (h)

The expected value of $\dot{\bar{x}}_Q \dot{\bar{x}}_Q^T$ is given by

$$E(\dot{\bar{x}}_Q \dot{\bar{x}}_Q^T) = E(C\bar{\theta}\bar{\theta}^T C^T) \tag{D14}$$

The components of C are substituted into (D14) and corrections in all directions are assumed to be equally likely. When this assumption and the one regarding the independence of errors in θ, ψ, and u are accounted for

$$E(\dot{\bar{x}}_Q \dot{\bar{x}}_Q^{\,T}) = \frac{1}{4}\begin{bmatrix} \left[v_{ms}(\sigma_\theta^2 + \sigma_\psi^2) + \sigma_u^2\right] & -v_{ms}\sigma_\psi^2 & 0 \\ -v_{ms}\sigma_\psi^2 & \left[v_{ms}(\sigma_\theta^2 + \sigma_\psi^2) + \sigma_u^2\right] & 0 \\ 0 & 0 & 2v_{ms}(\sigma_\theta^2) + \sigma_u^2 \end{bmatrix}$$

(D15)

where v_{ms} = trace of $E(\dot{\bar{x}}_G \dot{\bar{x}}_G^{\,T})$ computed before the velocity correction. See equation (D19). By definition

$$x_Q = \begin{bmatrix} 0 \\ \dot{\bar{x}}_Q \end{bmatrix}$$

where the zero represents a (3×1) null matrix.

Substituting in equation (D11) gives

$$S = \begin{bmatrix} 0 & 0 \\ 0 & E(\dot{\bar{x}}_Q \dot{\bar{x}}_Q^{\,T}) \end{bmatrix} \qquad \text{(D16)}$$

where the zeros represent 3×3 null matrices.

Once the value of R is known it can be used to compute the covariance matrix of the indicated velocity correction, $E(\dot{x}_G \dot{x}_G^{\,T})$. From equation (D6) and the definition of $\dot{x}_G$

$$\dot{\bar{x}}_G = -(A_2^{-1} A_1 \quad I)\hat{x}$$

so that

$$E(\dot{\bar{x}}_G \dot{\bar{x}}_G^{\,T}) = (A_2^{-1} A_1 \quad I)E(\hat{x}\hat{x}^T)(A_2^{-1} A_1 \quad I)^T \qquad \text{(D17)}$$

but it was shown that

$$E(\hat{x}\hat{x}^T) = (R-P) \qquad \text{(D18)}$$

so that

$$E(\dot{\bar{x}}_G\dot{\bar{x}}_G^T) = (A_2^{-1}\ A_1 \quad I)(R-P)(A_2^{-1}\ A_1 \quad I)^T \tag{D19}$$

The expression for $E(\dot{\bar{x}}_G\dot{\bar{x}}_G\ )$ is a 3×3 covariance matrix and v^2_{rms}, its trace or sum of the diagonal terms, is the mean square indicated velocity correction at the time the matrix is computed. To obtain the mean square of the velocity which would actually be added, one must add to v^2_{rms} the trace of S which is the mean square error in making a correction.

The square root of the sum of the first three terms in the diagonal of the P matrix gives the root-mean-square error, $\tilde{r}_{rms}$, in estimating position, while the last three give the corresponding quantity, $\tilde{v}_{rms}$, for velocity. The root-mean-square deviations, r_{rms} in position, and v_{rms} in velocity, are obtained from R by the same procedure. The remaining quantity presented in time history form is the rms position prediction error. If x_{EP} is the 3×1 column vector of errors in predicting end point position, the covariance matrix of prediction errors is $E(x_{EP}x_{EP}^T)$.

From equation (2) it is seen that

$$x_{EP} = (A_1 \quad A_2)\tilde{x} \tag{D20}$$

so that

$$E(x_{EP}x_{EP}^T) = (A_1 \quad A_2)E(\tilde{x}\tilde{x}^T)(A_1 \quad A_2)^T \tag{D21}$$

but

$$E(\tilde{x}\tilde{x}^T) = P$$

therefore

$$E(x_{EP}x_{EP}^T) = (A_1 \quad A_2)P(A_1 \quad A_2)^T \tag{D22}$$

The root-mean-square error, $\tilde{p}_{rms}$, in predicting position is the square root of the trace of the 3×3 matrix in (D22).

The computation of the rms deviation in radius of perigee and perilune given in table I is carried out at the time of reference periapsis. The radius of periapsis of the actual trajectory is

$$R_p = R_p\ (X,Y,Z,\dot{X},\dot{Y},\dot{Z}) \tag{D23}$$

The gradient, ∇R_p, is a 6×1 column vector of the partial derivatives. The total differential may be written as

$$dR_p = (\nabla R_p)^T dx \tag{D24}$$

where dx is the column vector of differentials of the Cartesian positions and velocities. If r_p is defined as the vector difference between the actual and reference periapsis vectors, it can be written as

$$r_p = (\nabla R_p)^T x \tag{D25}$$

The x here is the column vector of deviations between the actual and reference trajectories, and equation (D25) depends for its validity on the magnitude of x being relatively small. The variance of r_p is

$$E(r_p r_p^T) = E(\nabla R_p)^T(xx^T)(\nabla R_p)$$

which reduces to

$$E(r_p r_p^T) = (\nabla R_p)^T R(\nabla R_p)$$

REFERENCES

1. Smith, Gerald, L., Schmidt, Stanley F., and McGee, Leonard A.: Application of Statistical Filter Theory to the Optimal Estimation of Position and Velocity On Board a Circumlunar Vehicle. NASA TR R-135, 1962.

2. Kalman, R. E.: A New Approach to Linear Filtering and Prediction Problems. Trans. ASME, Ser. D, Jour. Basic Engr., vol. 82, no. 1, Mar. 1960, pp. 35-45.

3. Moulton, Forest Ray: An Introduction to Celestial Mechanics. Second ed., The Macmillan Co., 1959.

4. Stone, C. R., Johnson, Charles W., Smith, Fred B., Lee, E. Bruce, and Harvey, Charles A.: Time-Optimal Control of Linear Systems. Aero. Document R-ED6134, Minneapolis-Honeywell Regulator Co., Sept. 1, 1959.

TABLE I. - RESULTS AT PERIGEE - RMS VALUES

Condition	$R_{ref} - R_{act}$, km	Deviations		Uncertainties		Total corrective velocity, m/sec	rms fuel, % vehicle wt.
		r, km	v, m/sec	$\tilde{r}$, km	$\tilde{v}$, m/sec		
Standard trajectory	0.6	11.5	11.0	7.7	6.7	15.3	0.52
Perfect velocity correction	.1	5.6	5.2	4.6	4.0	12.0	.41
Observation errors ×5	1.2	33.5	29.9	26.9	23.5	22.3	.75
Injection errors ×5	.6	11.5	11.9	7.7	6.8	53.7	1.83
77 observations	1.1	26.4	23.8	15.0	13.2	20.0	.68

TABLE II. - RESULTS AT PERILUNE - RMS VALUES

Standard trajectory	1.7	4.8	1.2	0.8	0.08	9.9
Perfect velocity correction	.7	3.5	1.0	.7	.07	9.8
Observation errors ×5	3.5	13.5	3.0	3.5	.32	12.0
Injection errors ×5	1.7	4.9	4.1	.8	.08	43.3
77 observations	2.3	8.6	2.2	2.9	.27	13.3

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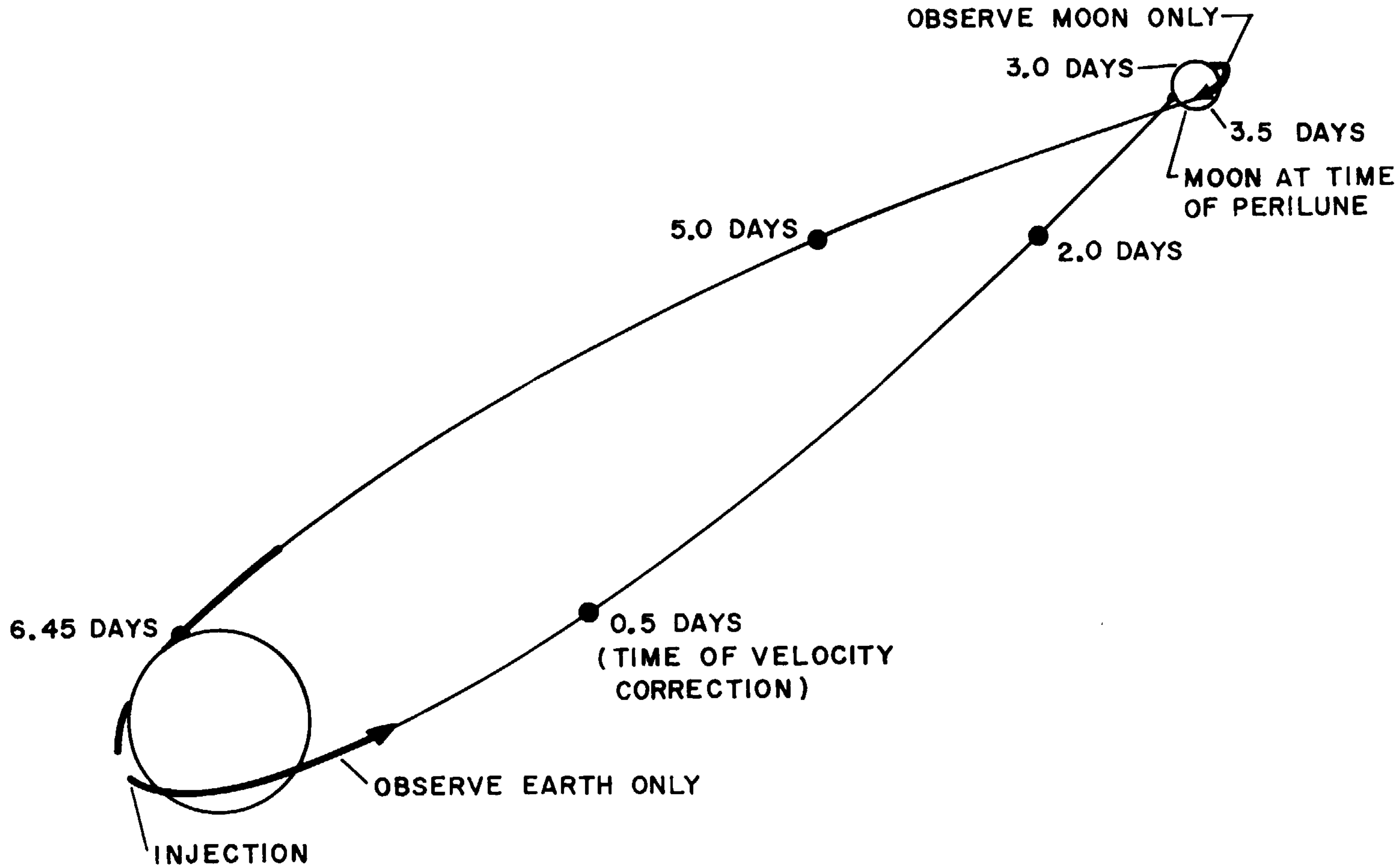

Figure 1.- Projection of reference trajectory into the orbital plane of the moon.

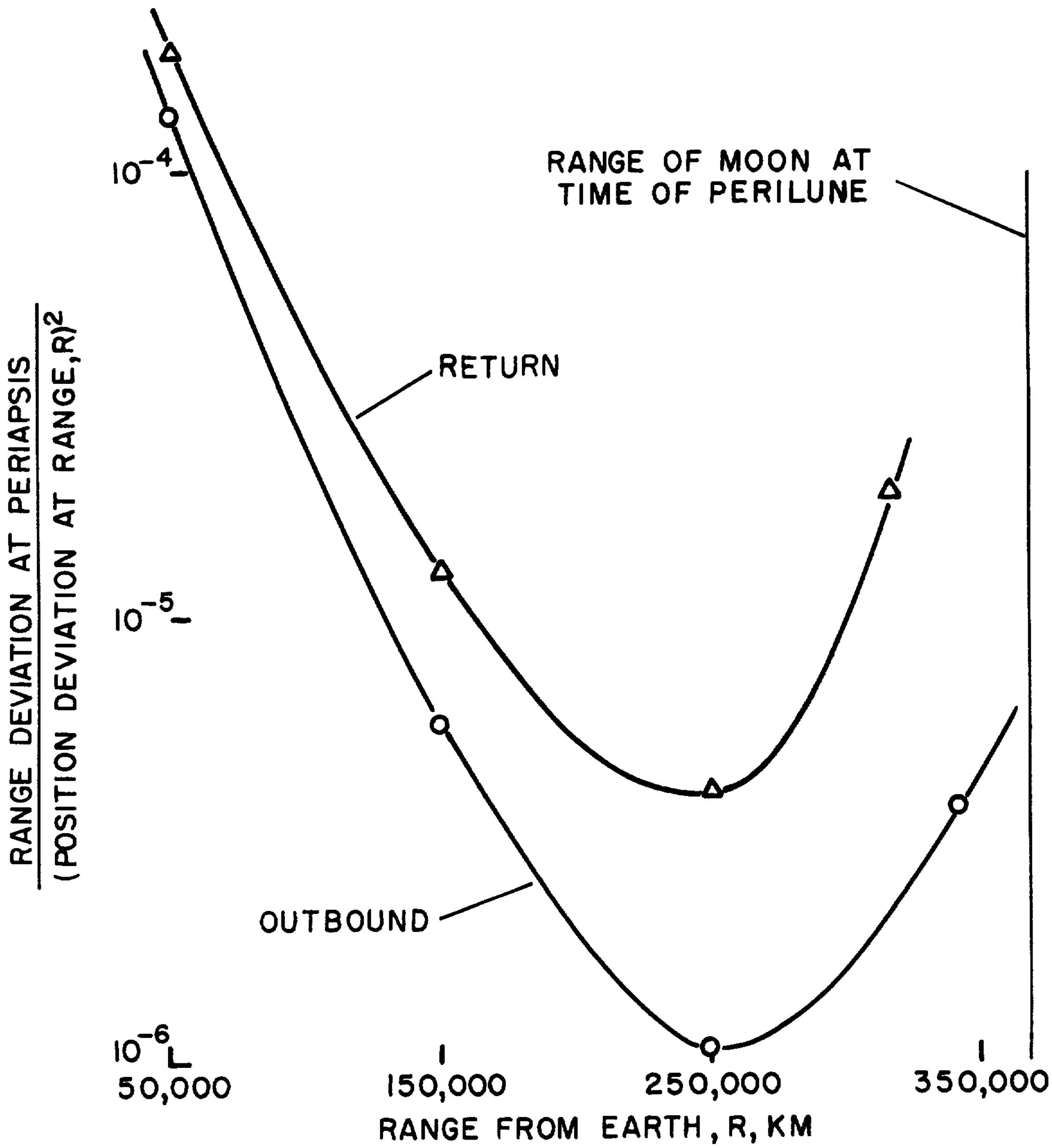

Figure 2.- Accuracy of linear prediction.

A
5
5
9

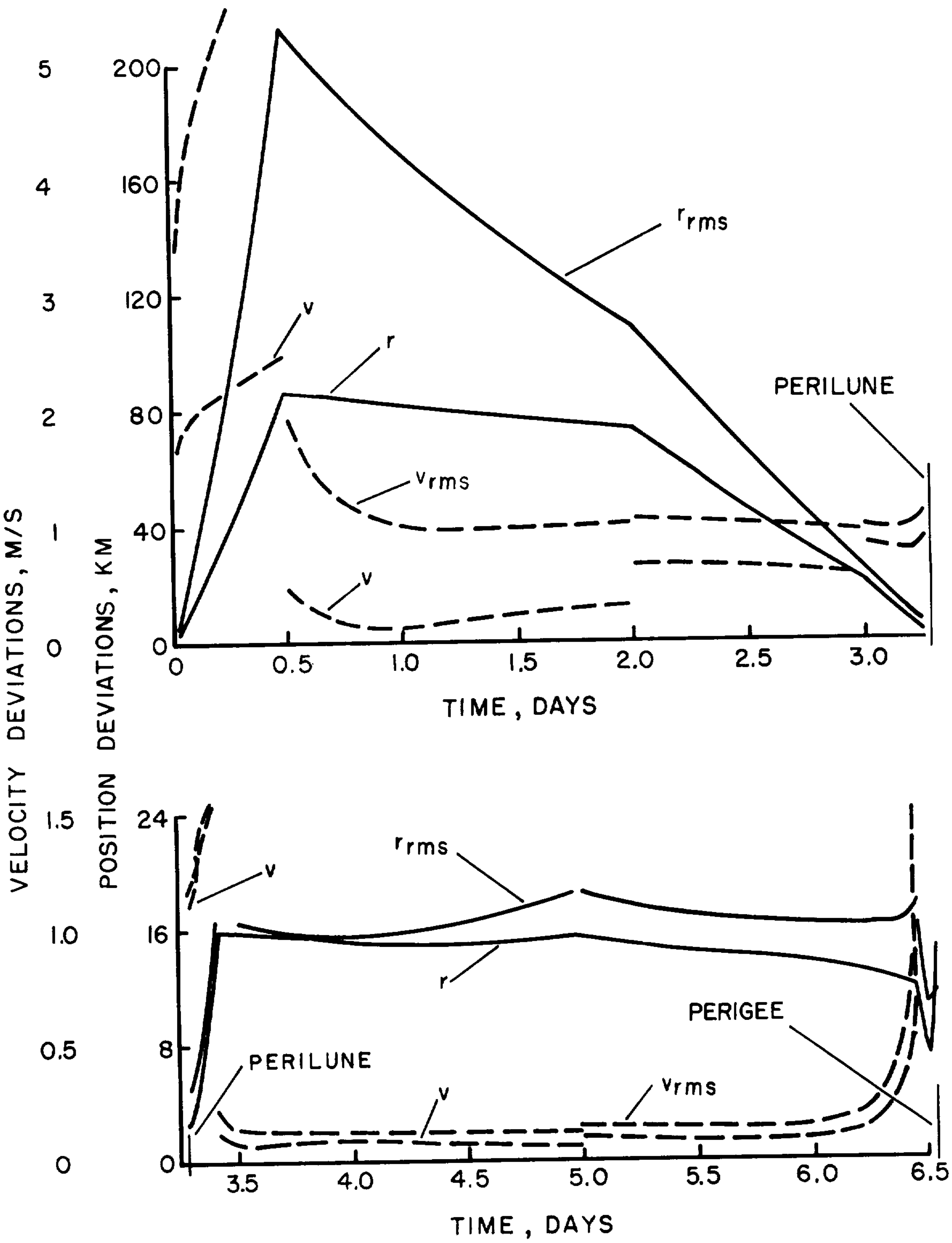

Figure 3.- Comparison of statistical data and one member of ensemble.

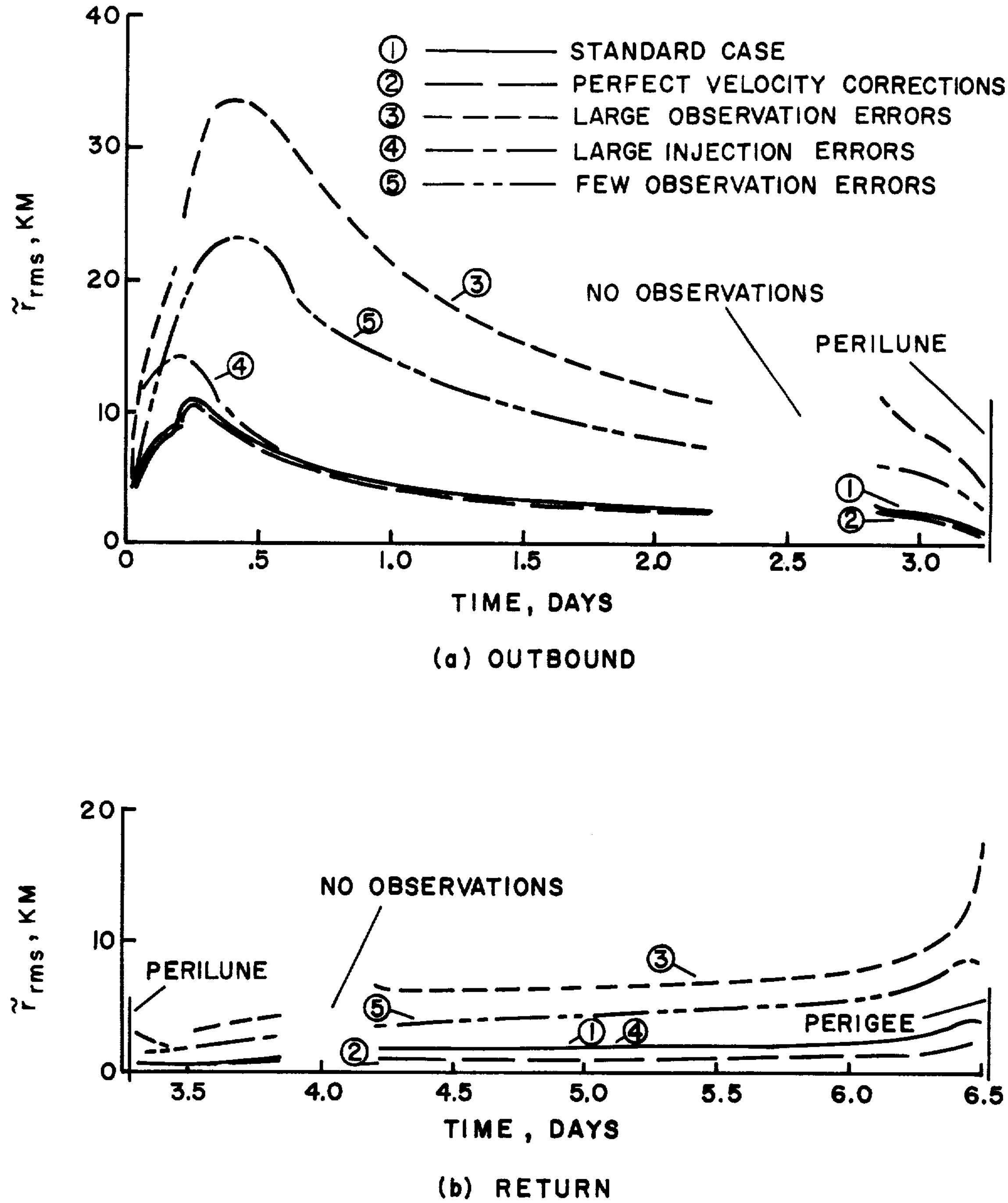

Figure 4.- Root-mean-square errors in estimation.

A
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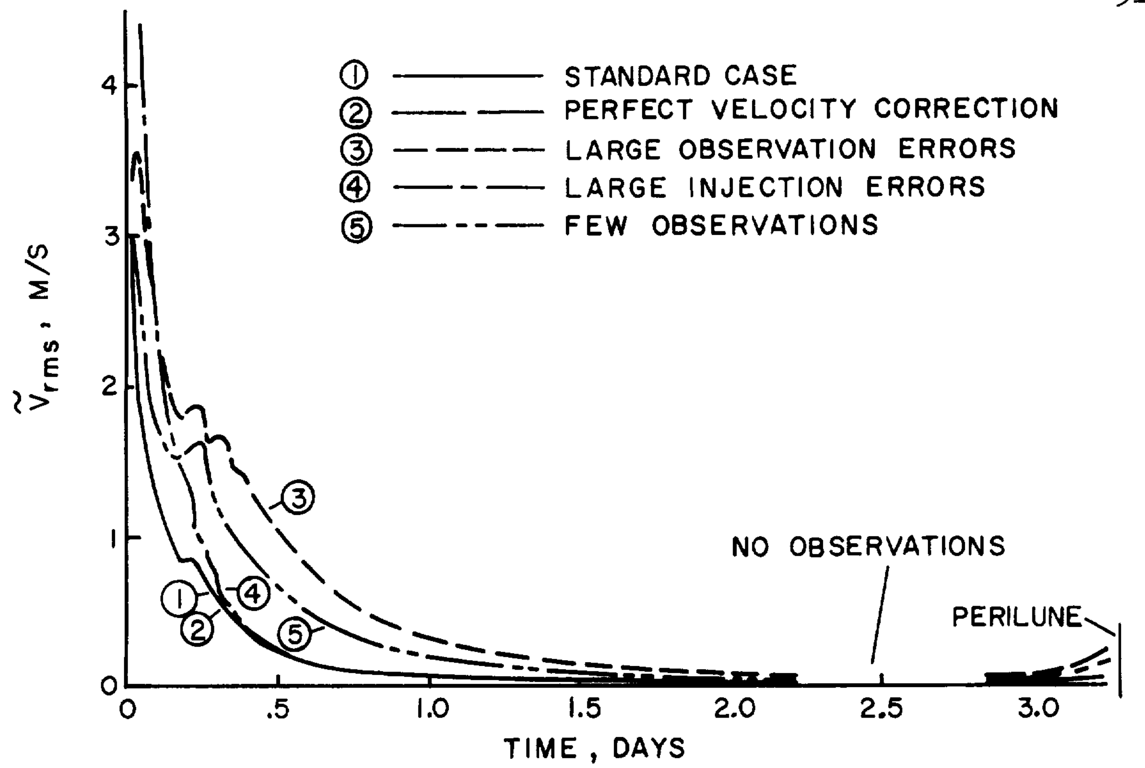
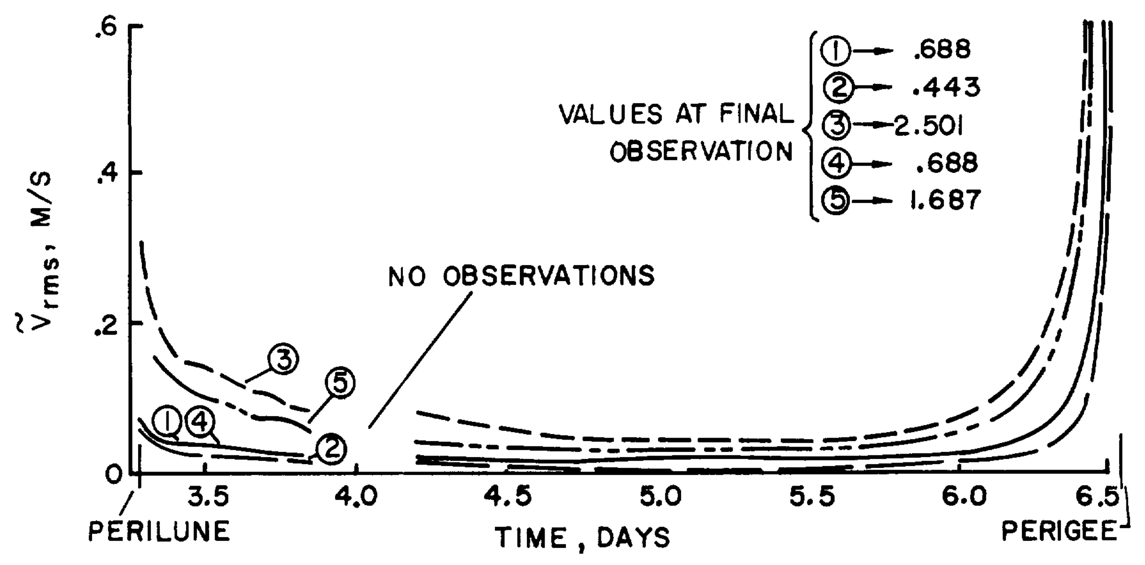
5
9



(c) OUTBOUND



(d) RETURN

Figure 4.- Concluded.

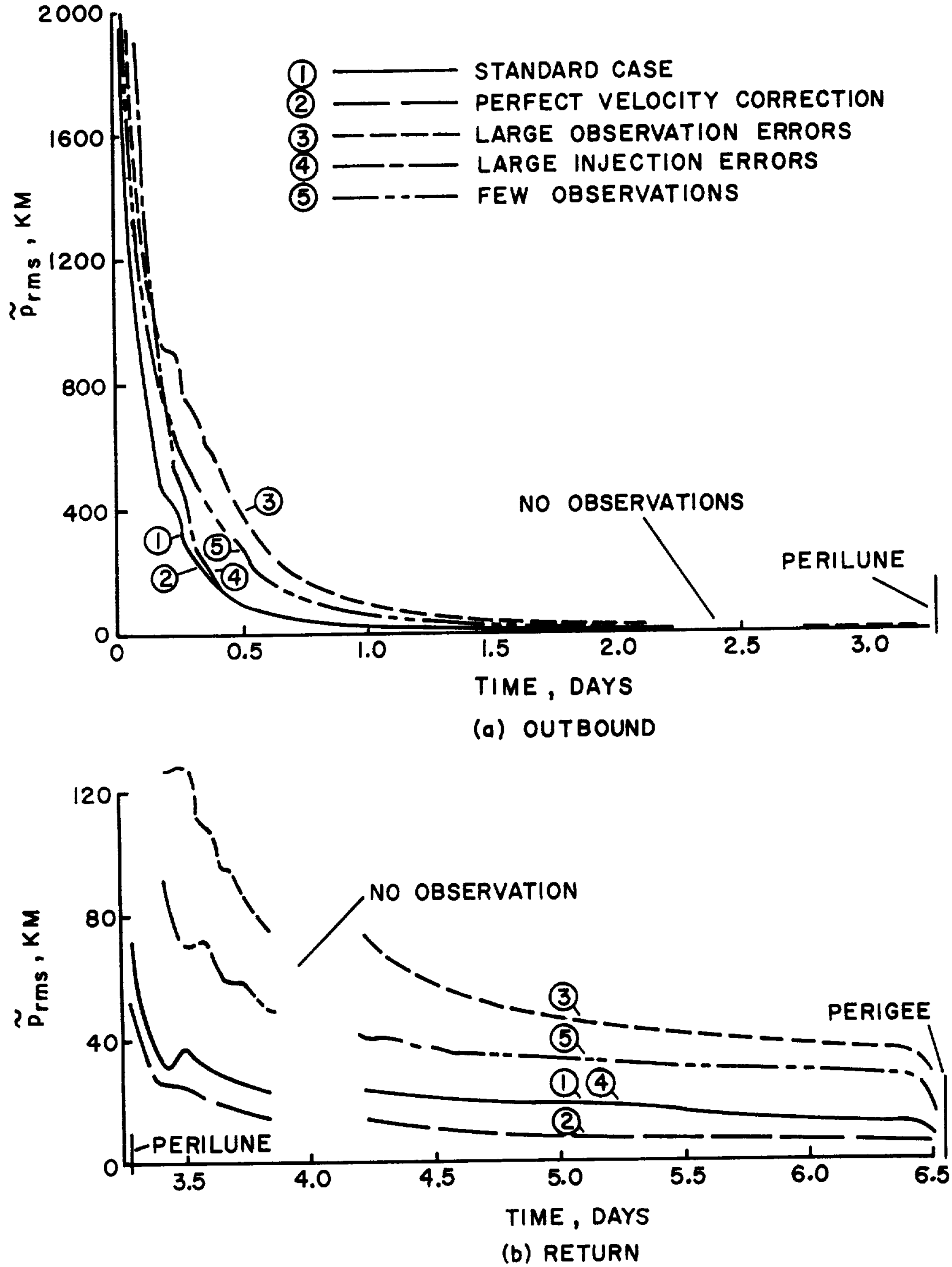


Figure 5.- Root-mean-square prediction errors.

A
5
5
9

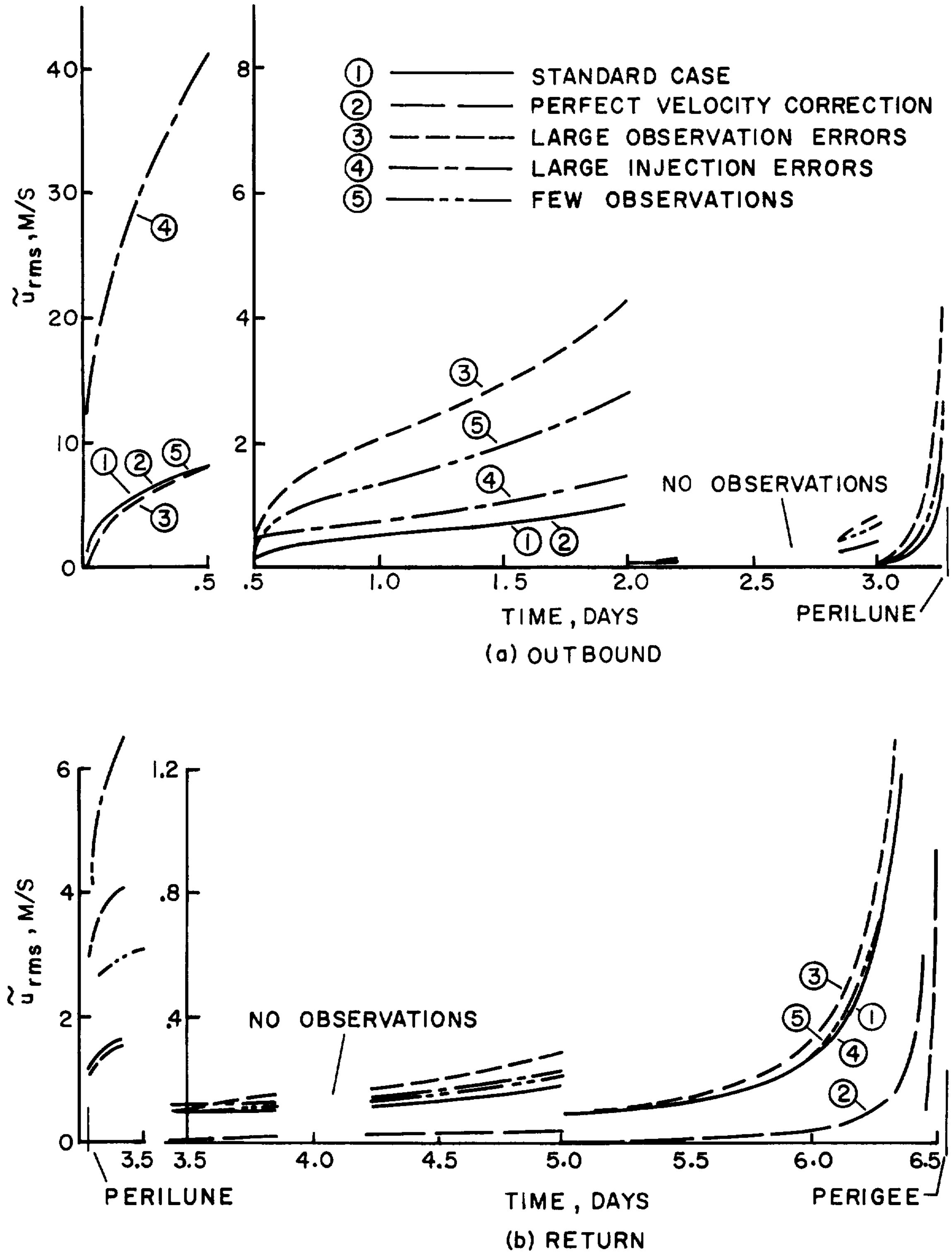
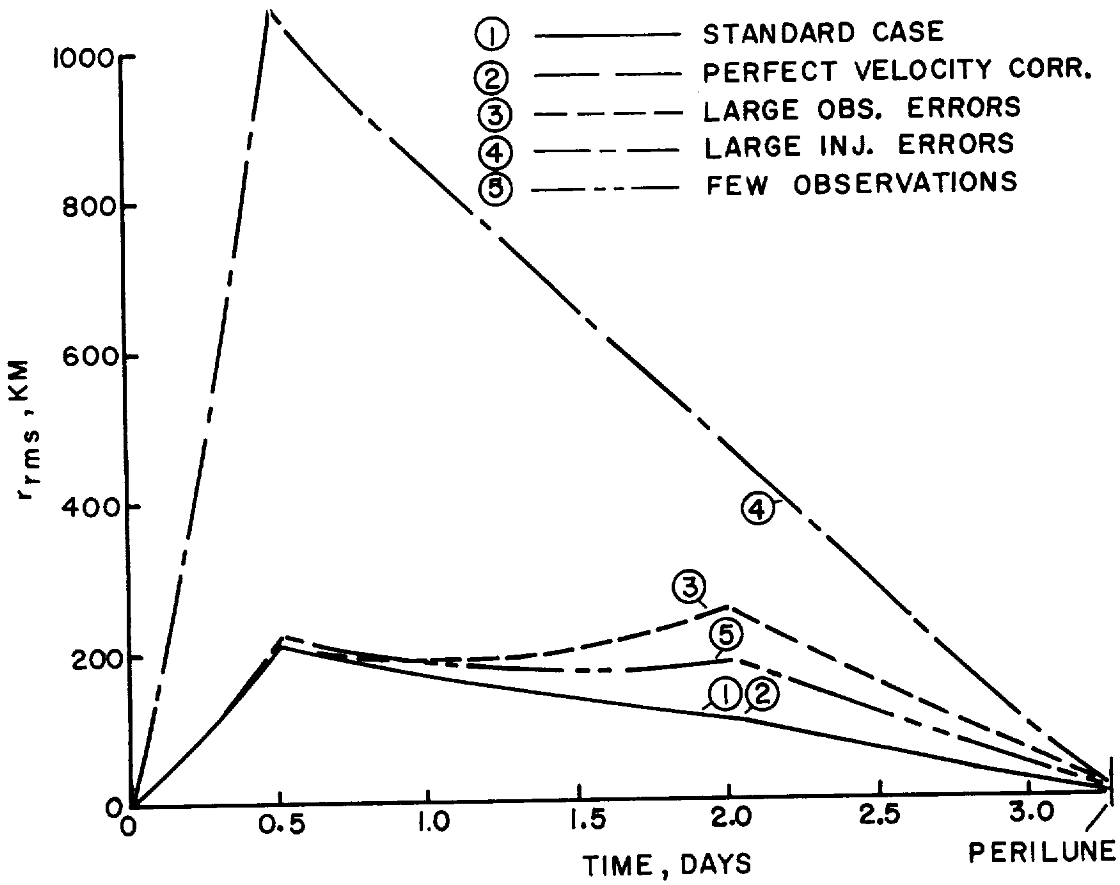
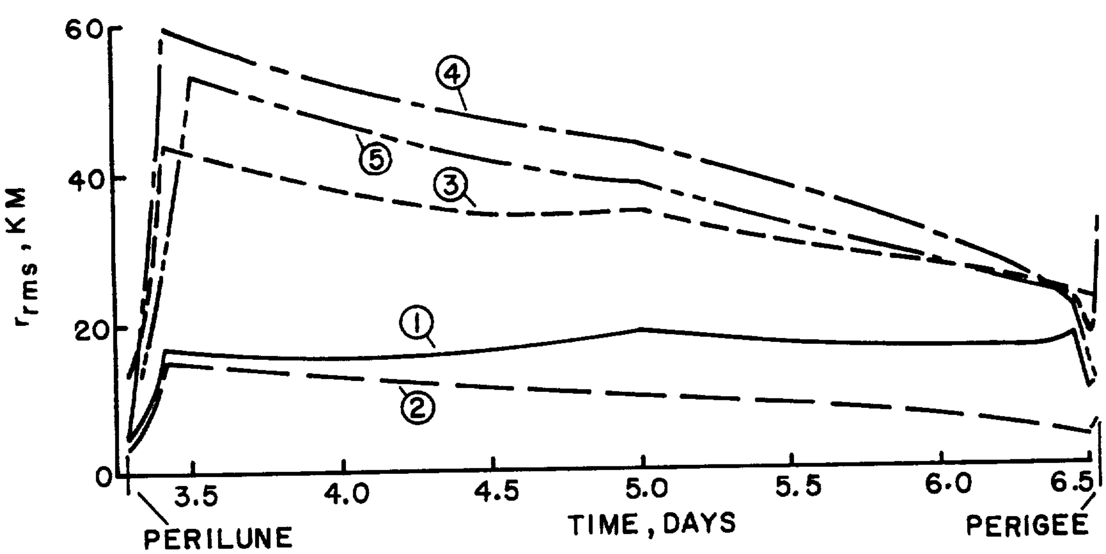


Figure 6.- Root-mean-square indicated velocity correction.



(a) OUTBOUND

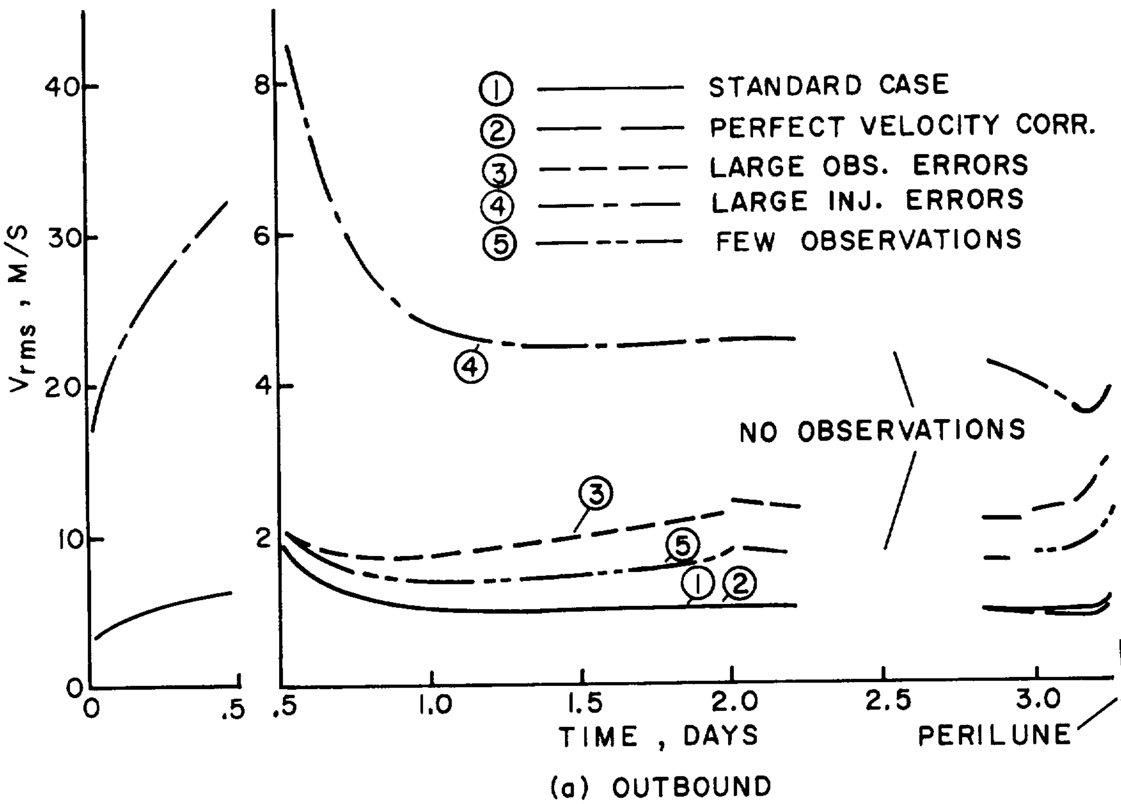


(b) RETURN

Figure 7.- Root-mean-square position deviation from reference trajectory.

A
5
5
9

A
5
5
9



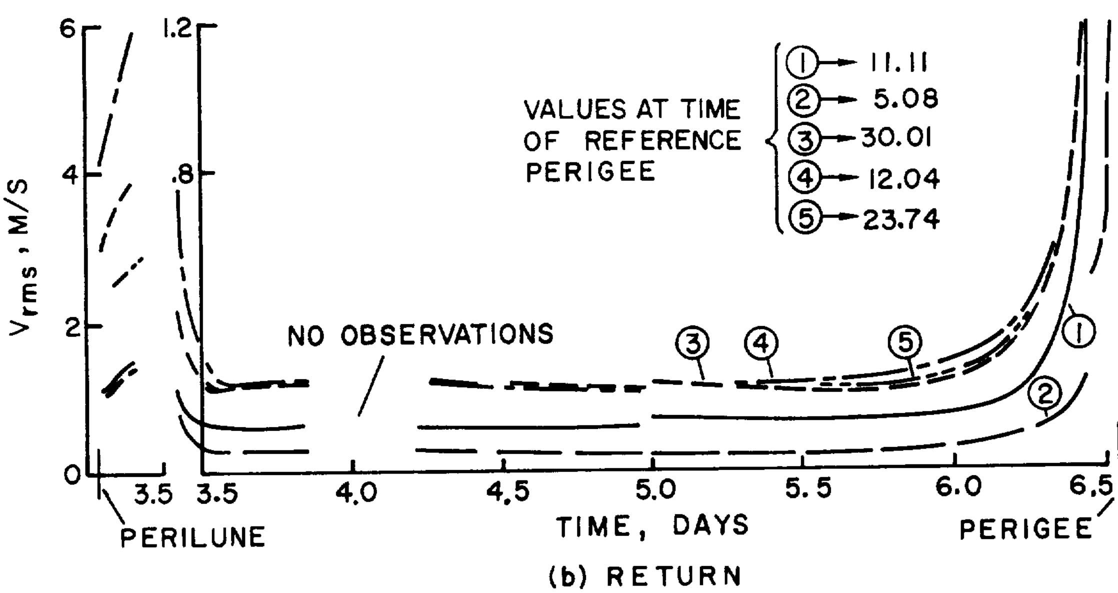


Figure 8.- Root-mean-square velocity deviation from reference trajectory.

NASA-Langley, 1962 A-559